Lecture 4

Vibration Suppression with Uncertain Load

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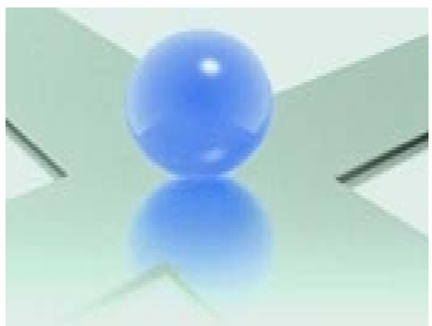

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1 *Highlights*

- **Models:**
  - **Conflicting.**
  - **Simplistic.**
  - **Incomplete.**
-

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  - **Conflicting.**
  - **Simplistic.**
  - **Incomplete.**
- **Data:**
  - **Random.**
  - **Biased, unknown correlations.**
  - **Subject to revision.**
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- **Models:**
  - **Conflicting.**
  - **Simplistic.**
  - **Incomplete.**
- **Data:**
  - **Random.**
  - **Biased, unknown correlations.**
  - **Subject to revision.**
- **Time:**
  - **Past may not reflect future.**
  - **Laws may change.**

§ The art of designing, deciding, planning:

Use the wrong model and data

to make the right decision

(when the right model is unknown).

§ Info-gap decision strategies:

- **Robust-satisficing: protect against uncertainty.**
-

§ Info-gap decision strategies:

- **Robust-satisficing:** **protect against uncertainty.**
- **Opportune-windfalling:** **exploit uncertainty.**

2 *DESIGN of a VIBRATING CANTILEVER*

2.1 *Design Problem*

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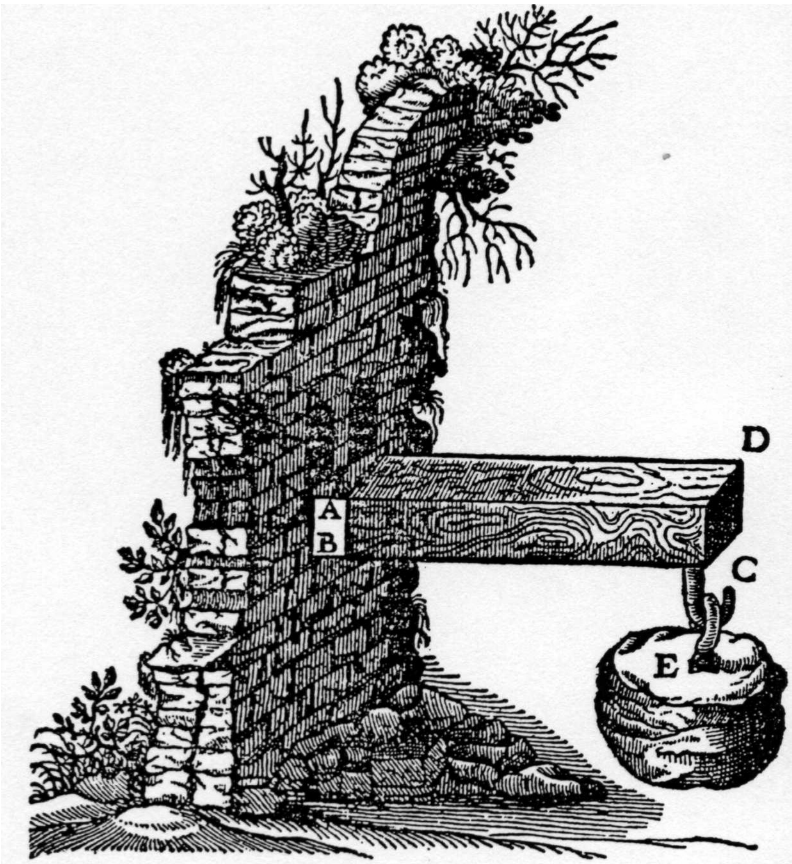

Fig. 17

§ Galileo's Cantilever.

§ The cantilever is a paradigm for:

- **Tall building.**
- **Radio tower.**
- **Crane.**
- **Airplane wing.**
- **Turbine blade.**
- **Diving board.**
- **Canon barrel.**
- **MEMS component.**
- **Atomic force microscope.**
- **etc.**

§ **Goal: Restrain vibration of cantilever from uncertain transient load.**

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§ **Two design concepts:**

- **Prevent vibration by stiffening the beam.**
- **Absorb vibration by dissipating energy.**

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from uncertain transient load.

§ **Two design concepts:**

- **Prevent vibration by stiffening the beam.**
- **Absorb vibration by dissipating energy.**

§ **Not mutually exclusive.**

§ **Relevant to different circumstances.**

2.2 *Robustness Function*

§ Three components of the analysis:

- **System model.**
- **Performance criterion.**
- **Uncertainty model.**

§ Simple system model:

Rigid vibration around clamped base.

$\theta(t)$ = **angle of deflection of beam.**

$u(t)$ = **moment of force at base.**

Equation of motion:

$$J\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k\theta = u(t)$$

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§ **Design variables:** $q = (c, k)$.

§ **System response:** $\theta(t, u, q)$.

§ **Uncertainty model.**

- **What we know about the load:**
  - **The nominal load, $\widetilde{u}(t)$.**
  - **The actual loads are transient:**
    - **May vary rapidly.**
    - **May deviate greatly from nominal load.**
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§ Uncertainty model.

- **What we know about the load:**
  - **The nominal load, $\widetilde{u}(t)$.**
  - **The actual loads are transient:**
    - **May vary rapidly.**
    - **May deviate greatly from nominal load.**
- **What we don't know about the load:**

  The actual realization, $u(t)$.

§ Info-gap model for uncertain transients:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \ \int_0^\infty \left[u(t) - \widetilde{u}(t) \right]^2 \, \mathrm{d}t \le h^2 \right\}, \qquad h \ge 0$$

h **= unknown horizon of uncertainty.**

$u(t)$ **= unknown actual load.**

$\widetilde{u}(t)$ **= known nominal load.**

§ **The performance criterion:**

Deflection must not exceed critical value:

$$|\theta(t, u, q)| \leq \theta_{\rm c}$$

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- **Satisfy vibration requirement.**
- **Maximize robustness.**
- **Don't try to minimize deflection.**

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- **Max acceptable horizon of uncertainty.**
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- **Max h at which $|\theta(t, u, q)| \le \theta_c$ guaranteed.**

$$\widehat{h}(q, \theta_c) = \max \left\{ h : \ \left(\max_{u \in \mathcal{U}(h, \widetilde{u})} \theta(t, u, q) \right) \le \theta_c \right\}$$

§ Derivation of robustness:

- **Info-gap model:**

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \int_0^{\infty} \left[u(t) - \widetilde{u}(t) \right]^2 \, \mathrm{d}t \le h^2 \right\}, \qquad h \ge 0$$

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- **System ($\theta(0) = \dot{\theta}(0) = 0$) and performance requirement:**

$$\begin{aligned} \theta_u(t) &= \int_0^t f(\tau) u(\tau) \, \mathrm{d}\tau, \qquad |\theta_u(t)| \le \theta_\mathrm{c} \\ &= \underbrace{\int_0^t f(\tau)[u(\tau) - \widetilde{u}(\tau)] \, \mathrm{d}\tau}_{A} + \int_0^t f(\tau) \widetilde{u}(\tau) \, \mathrm{d}\tau \end{aligned}$$

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- **Cauchy-Schwarz inequality:**

$$\left(\int a(t) b(t) \, \mathrm{d}t \right)^2 \le \int a^2(t) \, \mathrm{d}t \int b^2(t) \, \mathrm{d}t, \qquad \text{'=' if } a(t) \propto b(t)$$

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- **Use Cauchy-Schwarz inequality:**

$$A \le \sqrt{\int f^2(t) \mathrm{d}t} \underbrace{\sqrt{\int [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t}}_{h}$$

-

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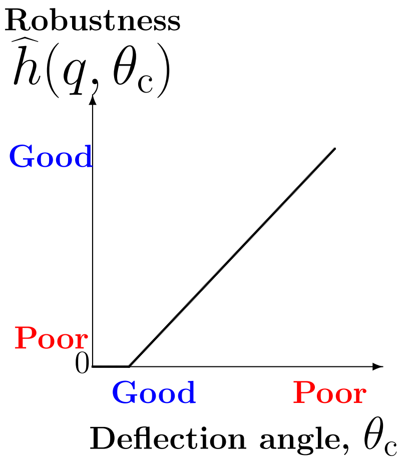
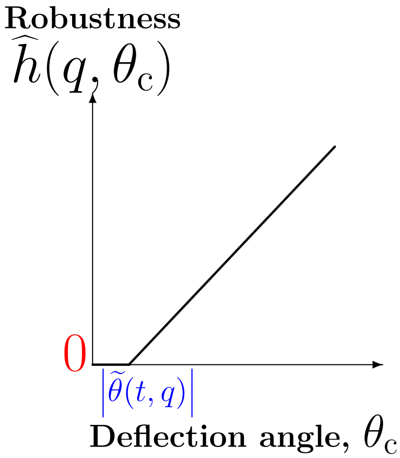
- **Thus:**

$$\widehat{h}(q,\theta_\mathrm{c}) = \frac{\theta_\mathrm{c} - \left|\widetilde{\theta}(t,q)\right|}{\sqrt{\int_0^t f^2(\tau,q)\,\mathrm{d}\tau}} \quad \text{if } \theta_\mathrm{c} \ge \left|\widetilde{\theta}(t,q)\right|$$



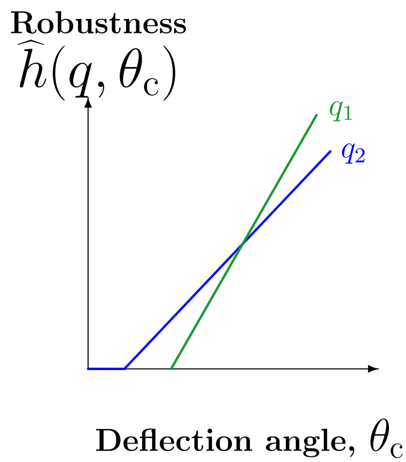
§ Trade-off: robustness vs. performance.

$$\widehat{h}(q, \theta_c) = \begin{cases} \dfrac{\theta_c - \left|\widetilde{\theta}(t, q)\right|}{\sqrt{\int_0^t f^2(\tau, q)\, d\tau}} & \textbf{if } \theta_c \geq \left|\widetilde{\theta}(t, q)\right| \\ 0 & \textbf{else} \end{cases}$$

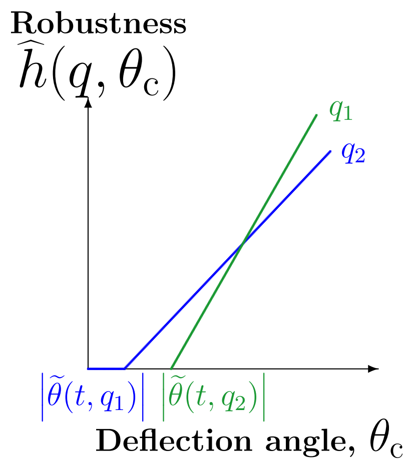


§ **Zeroing: Estimated performance has no robustness:**

$$\widehat{h}(q, \theta_{\rm c}) = 0 \quad \textbf{if} \quad \theta_{\rm c} = \left|\widetilde{\theta}(t, q)\right|$$



§ **Two designs: q_1 and q_2.**

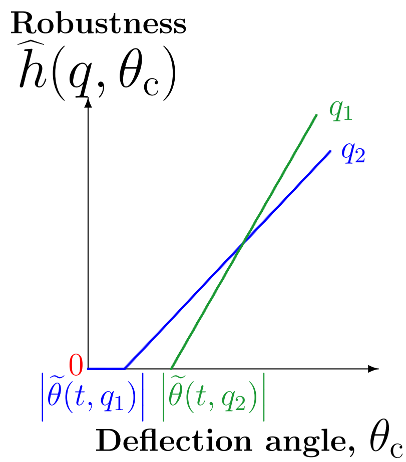


§ Best-model preference:

$$\left|\widetilde{\theta}(t, q_1)\right| < \left|\widetilde{\theta}(t, q_2)\right|$$

implies:

$$q_1 \succ q_2$$



§ Best-model preference has no robustness:

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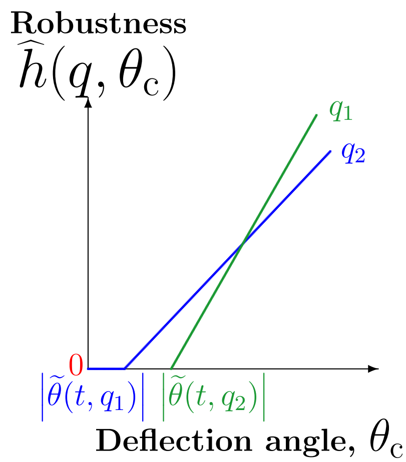

§ Best-model preference has no robustness:

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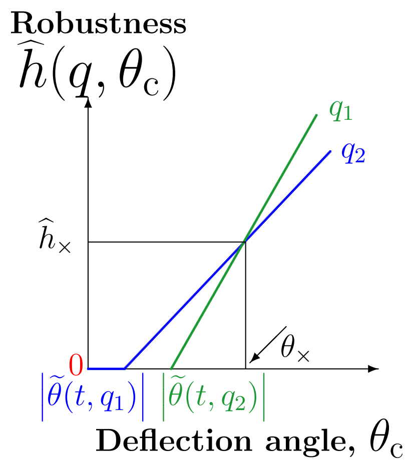
Thus

$$\left|\widetilde{\theta}(t, q_1)\right| < \left|\widetilde{\theta}(t, q_2)\right|$$

is not a good basis for preferring q_1.



§ **Best-model preference:** $q_1 \succ q_2$.

§ **Preference reversal:** $q_2 \succ q_1$

if $\theta_\times$ is adequate.

2.3 *Numerical Example*

§ Nominal input: Time-varying load.

- **Estimated input, $\widetilde{u}(t)$, is square:**

$$\widetilde{u}(t) = \begin{cases} \widetilde{u}_o, & 0 \le t \le T \\ 0, & t > T \end{cases}$$

-

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§ **Robust-satisficing design strategy:**

- **Satisfy vibration requirement.**
- **Maximize robustness.**
- **Don't try to optimize (minimize) deflection.**

§ Design variables:

- $q = (c, k)$: **damping and stiffness.**
- $\zeta =$ **dimensionless damping.**
- $\omega =$ **natural frequency.**

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- $q = (c, k)$: **damping and stiffness.**
- $\zeta =$ **dimensionless damping.**
- $\omega =$ **natural frequency.**

§ Consider 6 designs:

- $\omega =$ **1, 3, 4.** $\zeta =$ **0.01.**
- $\zeta =$ **0.03, 0.3, 0.5.** $\omega =$ **1.**

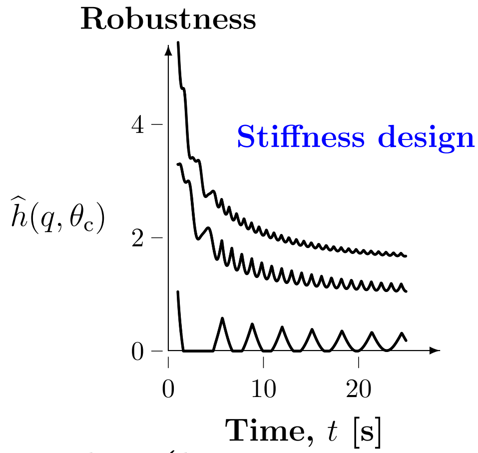


Figure 1: ω = **1, 3 and 4 (bottom to top).** ζ = **0.01.**

§ Variable stiffness; low damping:

- $\widehat{h}$ **oscillates and decreases over time.**
- **Low stiffness (**$\omega = 1$**):** $\widehat{h}$ **periodically zero.**
- **Moderate and high stiffness (**$\omega = 3, 4$**):** $\widehat{h}$ **oscillates but positive.**
- **Large** $\widehat{h}$ **at** $t < T$**.**

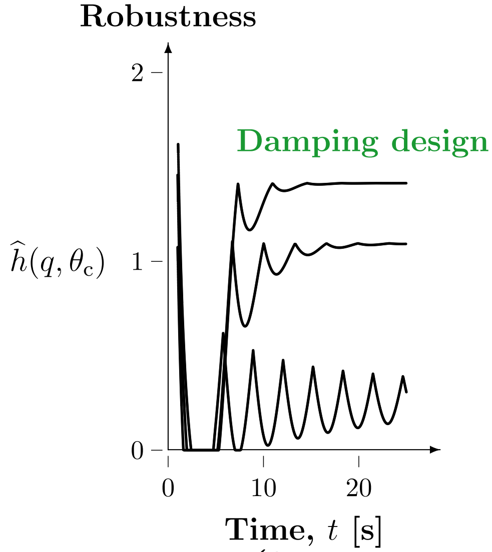
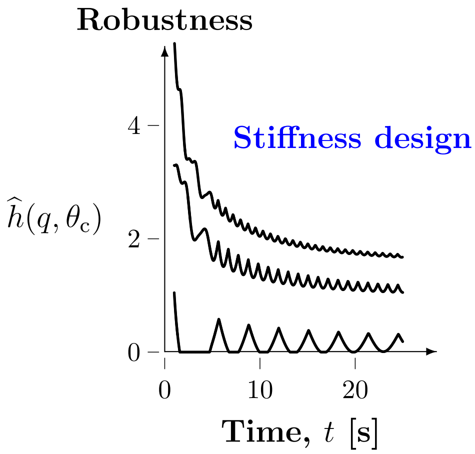
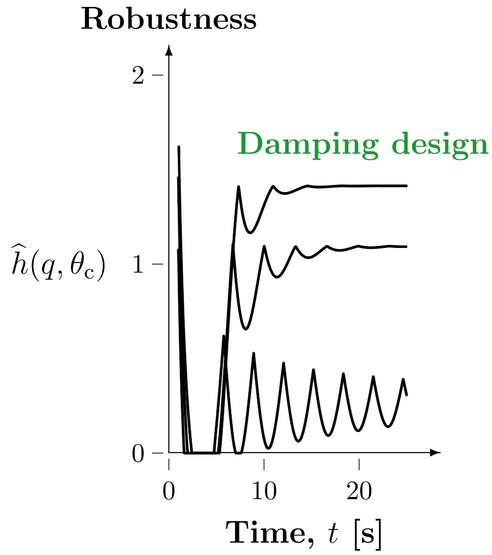


Figure 2: ζ = **0.03, 0.3, 0.5 (bottom to top).** ω = **1.**

§ Variable damping; low stiffness:

- **Low damping: same as before.**
- **Large damping ($\zeta = 0.3$ or 0.5):**
  - $\widehat{h}$ **small for** $t \leq T$.
  - $\widehat{h}$ **large and constant for** $t > T$.





§ Design implications:

Time frame determines design concept:

- $t < T$: **stiffness design.**
- $t > T$: **dissipation design.**
- $t > 0$: **combined stiff. & dissip. design.**

2.4 *Opportuneness Function*

§ **Windfall reward: angular deflection θ_{w} much less than critical requirement, θ_{c}:**

$$\theta_{\mathrm{w}} \ll \theta_{\mathrm{c}}$$

§ Opportuneness of design q:

- **Minimum promising horizon of uncertainty.**
- **Minimum h at which $|\theta(t,u,q)| \le \theta_{\rm w}$ possible.**

$$\widehat{\beta}(q,\theta_{\rm w}) = \min\left\{h: \ \left(\min_{u\in\mathcal{U}(h,\tilde{u})} \theta(t,u,q)\right) \le \theta_{\rm w}\right\}$$

§ **Compare opportuneness and robustness.**

§ **Opportuneness of design q:**

- **Minimum promising horizon of uncertainty.**
- **Minimum h at which windfall possible.**

$$\widehat{\beta}(q,\theta_{\mathrm{w}}) = \min\left\{h: \ \left(\min_{u\in\mathcal{U}(h,\tilde{u})}\theta(t,u,q)\right) \le \theta_{\mathrm{w}}\right\}$$

§ **Robustness of design q:**

- **Maximum acceptable horizon of uncertainty.**
- **Maximum h at which critical requirement guaranteed.**

$$\widehat{h}(q,\theta_{\mathrm{c}}) = \max\left\{h: \ \left(\max_{u\in\mathcal{U}(h,\tilde{u})}\theta(t,u,q)\right) \le \theta_{\mathrm{c}}\right\}$$

§ Opportuneness function:

$$\widehat{\beta}(q,\theta_{\mathrm{w}}) = \begin{cases} \dfrac{\left|\widetilde{\theta}(t,q)\right| - \theta_{\mathrm{w}}}{\sqrt{\int_0^t f^2(\tau,q)\,\mathrm{d}\tau}} & \textbf{if } \theta_{\mathrm{w}} \le \left|\widetilde{\theta}(t,q)\right| \\ 0 & \textbf{else} \end{cases}$$

§ **Compare opportuneness to robustness:**

$$\widehat{\beta}(q, \theta_{\mathrm{w}}) = -\widehat{h}(q, \theta_{\mathrm{c}}) + \frac{\theta_{\mathrm{c}} - \theta_{\mathrm{w}}}{\sqrt{\int_0^t f^2(\tau)\,\mathrm{d}\tau}}$$

§ **Antagonism or sympathy of immunity functions?**

- $\widehat{h}(q, \theta_{\mathrm{c}})$: **Bigger is better.**
- $\widehat{\beta}(q, \theta_{\mathrm{w}})$: **Big is bad.**
-

§ **Compare opportuneness to robustness:**

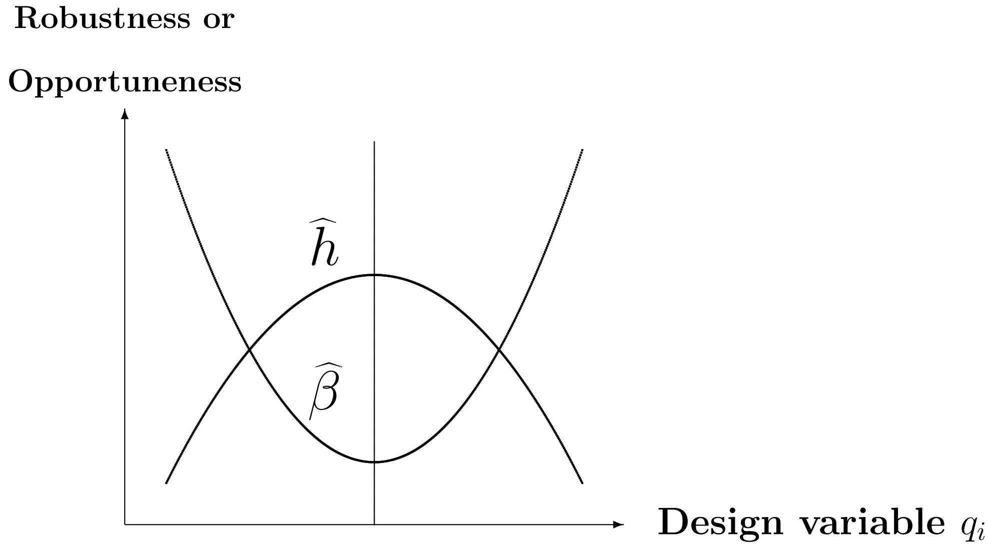
$$\widehat{\beta}(q, \theta_{\rm w}) = -\widehat{h}(q, \theta_{\rm c}) + \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}}$$

§ **Antagonism or sympathy of immunity functions?**

- $\widehat{h}(q, \theta_{\rm c})$: **Bigger is better.**
- $\widehat{\beta}(q, \theta_{\rm w})$: **Big is bad.**
- $q = (\omega, \zeta)$ **design vector.**
- $\widehat{\beta}(q, \theta_{\rm w})$ **and** $\widehat{h}(q, \theta_{\rm c})$ **are sympathetic if they can be improved simultaneously.**
- **Antagonistic otherwise.**

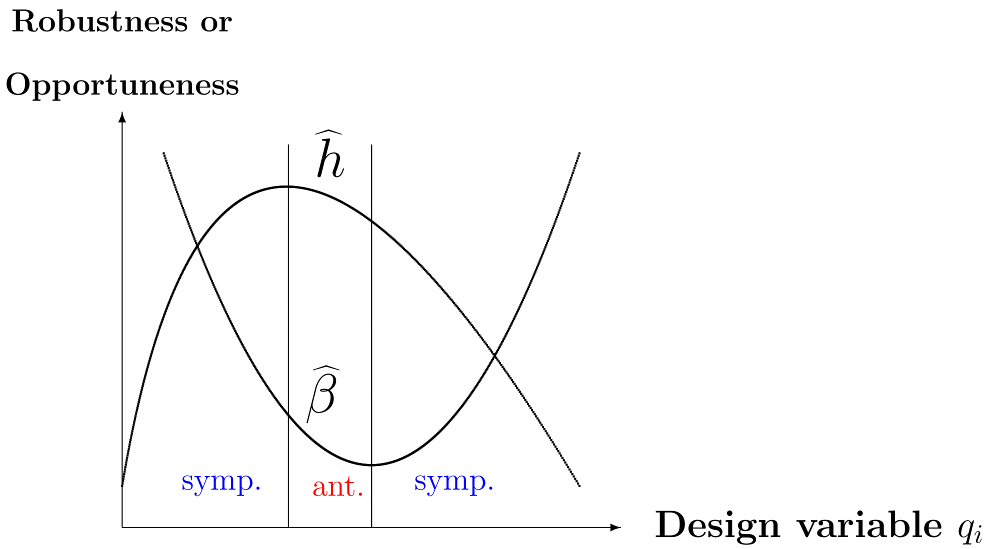
§ $\widehat{h}$ and $\widehat{\beta}$ always sympathetic

if and only if their optima coincide.



§ "Usually" this will not happen.

§ Usually:



2.5 *Summary of Vibrating Cantilever Example*

§ Use load-estimate to choose design.

§

§ **Use load-estimate to choose design.**

§ **Load-estimates err: info-gaps.**

Hence: require robustness.

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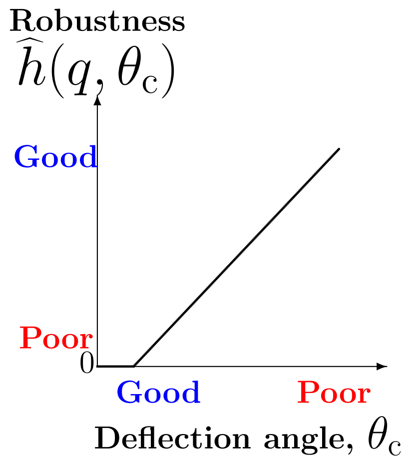
Hence: require robustness.

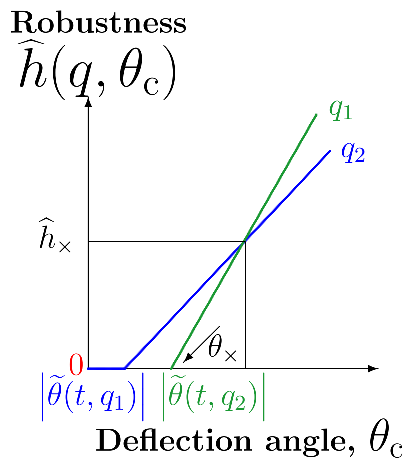
§ **Load-estimate design: zero robustness.**

§ **Robustness trades-off against performance.**



§ Robustness curves may cross:

Preference reversal.



§ Robust-satisficing design strategy:

- **Satisfice vibration requirement.**
- **Maximize robustness to failure.**

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§ Opportune-windfalling design strategy:

- **Seek wonderful vibration outcome.**
- **Minimize immunity to windfall.**

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- **Satisfice vibration requirement.**
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§ Robustness and opportuneness may be sympathetic or antagonistic.

3 *SUMMARY*

§ **Models:**

Attributes of model correspond to attributes of reality.

§ **Model-based decision:**

Adapt decision to attributes of model.

§ **Optimization:**

Use best model to choose decision with best outcome.

§

§ **Models:**

Attributes of model correspond to attributes of reality.

§ **Model-based decision:**

Adapt decision to attributes of model.

§ **Optimization:**

Use best model to choose decision with best outcome.

§ **But there is deep uncertainty:**

- **Randomness: structured uncertainty.**
- **Info-gaps: surprises, ignorance.**

§ Fallacy of optimal model-based decision:

- **Severe uncertainty:**
  - **Best model errs seriously.**
  - **Some model attributes are correct.**
  - **Some model attributes err greatly.**
- **Best-model optimization**
  - **exploits all model attributes to extreme.**
  - **Vulnerable to model error.**

§ Resolution: Info-gap decision theory.

- **Satisfice performance.**
- **Optimize robustness to uncertainty.**
- **Model and manage sur$^{\mathbf{p}}$rises.**

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§ **Resolution: Info-gap decision theory.**

- **Satisfice performance.**
- **Optimize robustness to uncertainty.**
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§ **Robust-satisficing syllogism:**

- **Adequate performance is necessary.**
- **More reliable adequate performance is better than less reliable adeq. perf.**
- **Thus maximum reliability is best.**

§

§ Resolution: Info-gap decision theory.

- **Satisfice** performance.
- **Optimize** robustness to uncertainty.
- Model and manage **sur$^{\mathbf{p}}$rises**.

§ Robust-satisficing syllogism:

- Adequate performance is necessary.
- More reliable adequate performance is better than less reliable adeq. perf.
- Thus maximum reliability is best.

§ Opportune windfalling:

Exploit uncertain opportunities.

§ **Sources: http://info-gap.technion.ac.il**

§ **Applications of info-gap theory:**

- **Monetary economics.**
- **Financial stability.**
- **Public policy and regulation.**
- **Climate change.**
- **Engineering design.**
- **Biological conservation.**
- **Sampling, assay design.**
- **Medical decision making.**
- **Fault detection and diagnosis.**
- **Project management.**
- **Homeland security.**
- **Statistical hypothesis testing.**