94. **Allocation of scarce resource** (based on exam in 036057, 16.1.2017), (p.339). Consider allocation of a scarce resource, such as time or money, among a number of different items. Given $N > 1$ items and a total resource budget R, let r_n denote the allocation to item n, for $n = 1, \ldots, N$, where $r_n \geq 0$. The benefit resulting from allocating r_n to item n is $r_n b_n$ where the benefit per unit allocation, b_n, is uncertain. The total benefit is $B = \sum_{n=1}^{N} r_n b_n$, and we require that the total benefit be no less than the critical value B_c.

(a) The benefit per unit allocation is estimated as $\widetilde{b}_n \pm s_n$, but it may be either less or more, where $\widetilde{b}_n > 0$ and $s_n > 0$ are known. The info-gap model for uncertainty is:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b_n - \widetilde{b}_n}{s_n} \right| \leq h,\ n = 1, \ldots, N \right\}, \quad h \geq 0 \tag{453}$$

Derive an explicit algebraic expression for the robustness function.

(b) Let $\widetilde{b}$ and s denote the vectors of estimated benefits per unit allocation, $\widetilde{b}_n$, and error weights, s_n, respectively. Consider two different vectors of allocations $r = (r_1, \ldots, r_N)$ and $\rho = (\rho_1, \ldots, \rho_N)$. These allocations satisfy the following relations:

$$r^T \widetilde{b} > \rho^T \widetilde{b} \tag{454}$$

$$\frac{r^T \widetilde{b}}{r^T s} < \frac{\rho^T \widetilde{b}}{\rho^T s} \tag{455}$$

What is an intuitive interpretation of these relations? Specifically, how do they reflect a dilemma facing the decision maker? Using the answer to part 94a, derive an explicit algebraic expression for the values of critical benefit, B_c, for which allocation r is robust-preferred over allocation ρ.

(c) Return to the basic formulation of the problem, prior to part 94a, and consider two different programs within which the resource can be allocated. Program 1 has nominal predicted total benefit B_1 which is a known positive number. However, the actual benefits are uncertain and the robustness function for allocation vector r in program 1 is known and finite for all values of B_c. Program 2 has exactly known benefits, and the total benefit is guaranteed to be B_2 for the same allocation vector, r. However, $B_2 < B_1$. Derive an explicit algebraic expression for the values of critical benefit, B_c, for which program 1 is robust-preferred over program 2.

(d) Return to the basic formulation of the problem, prior to part 94a, and consider the following ellipsoid-bound info-gap model for uncertainty in the benefit vector:

$$\mathcal{U}(h) = \left\{ b : (b - \widetilde{b})^T W^{-1} (b - \widetilde{b}) \leq h^2 \right\}, \quad h \geq 0 \tag{456}$$

where W is a real, symmetric, positive definite $N \times N$ matrix. Derive an explicit algebraic expression for the robustness function.

(e) Suppose that the total benefit, B, is an exponentially distributed random variable, whose probability density function is:

$$p(B) = \lambda \mathrm{e}^{-\lambda B}, \quad B \geq 0 \tag{457}$$

What is the probability that the total benefit exceeds the critical value B_c?

(f) Continuing part 94e, suppose that you require that the probability of exceeding the critical benefit, B_c, must be no less than the critical probability P_c. However, the critical benefit,

B_c, is uncertain (you don't really know what you need). Use the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ B_c : \left| \frac{B_c - \widetilde{B}_c}{\widetilde{B}_c} \right| \le h \right\}, \quad h \ge 0 \tag{458}$$

Derive an explicit algebraic expression for the robustness function for satisfying the probabilistic requirement.

(g) Repeat part 94a with the following info-gap model:

$$\mathcal{U}(h) = \left\{ b : \left(b - \widetilde{b} \right)^T W^{-1} \left(b - \widetilde{b} \right) \le h^2 \right\}, \quad h \ge 0 \tag{459}$$

where W is a real, symmetric positive definite matrix. W and $\widetilde{b}$ are known. Derive an explicit algebraic expression for the robustness function.