94. **Allocation of scarce resource** (based on exam in 036057, 16.1.2017), (p.313). Consider allocation of a scarce resource, such as time or money, among a number of different items. Given $N > 1$ items and a total resource budget R, let r_n denote the allocation to item n, for $n = 1, \ldots, N$, where $r_n \ge 0$. The benefit resulting from allocating r_n to item n is $r_n b_n$ where the benefit per unit allocation, b_n, is uncertain. The total benefit is $B = \sum_{n=1}^{N} r_n b_n$, and we require that the total benefit be no less than the critical value $B_{\rm c}$.

(a) The benefit per unit allocation is estimated as $\widetilde{b}_n \pm s_n$, but it may be either less or more, where $\widetilde{b}_n > 0$ and $s_n > 0$ are known. The info-gap model for uncertainty is:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b_n - \widetilde{b}_n}{s_n} \right| \le h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{453}$$

Derive an explicit algebraic expression for the robustness function.

(b) Let $\widetilde{b}$ and s denote the vectors of estimated benefits per unit allocation, $\widetilde{b}_n$, and error weights, s_n, respectively. Consider two different vectors of allocations $r = (r_1, \ldots, r_N)$ and $\rho = (\rho_1, \ldots, \rho_N)$. These allocations satisfy the following relations:

$$r^T \widetilde{b} \; > \; \rho^T \widetilde{b} \tag{454}$$

$$\frac{r^T \widetilde{b}}{r^T s} \; < \; \frac{\rho^T \widetilde{b}}{\rho^T s} \tag{455}$$

What is an intuitive interpretation of these relations? Specifically, how do they reflect a dilemma facing the decision maker? Using the answer to part 94a, derive an explicit algebraic expression for the values of critical benefit, $B_{\rm c}$, for which allocation r is robust-preferred over allocation ρ.

(c) Return to the basic formulation of the problem, prior to part 94a, and consider two different programs within which the resource can be allocated. Program 1 has nominal predicted total benefit B_1 which is a known positive number. However, the actual benefits are uncertain and the robustness function for allocation vector r in program 1 is known and finite for all values of $B_{\rm c}$. Program 2 has exactly known benefits, and the total benefit is guaranteed to be B_2 for the same allocation vector, r. However, $B_2 < B_1$. Derive an explicit algebraic expression for the values of critical benefit, $B_{\rm c}$, for which program 1 is robust-preferred over program 2.

(d) Return to the basic formulation of the problem, prior to part 94a, and consider the following ellipsoid-bound info-gap model for uncertainty in the benefit vector:

$$\mathcal{U}(h) = \left\{ b : \ (b - \widetilde{b})^T W^{-1} (b - \widetilde{b}) \le h^2 \right\}, \quad h \ge 0 \tag{456}$$

where W is a real, symmetric, positive definite $N \times N$ matrix. Derive an explicit algebraic expression for the robustness function.

(e) Suppose that the total benefit, B, is an exponentially distributed random variable, whose probability density function is:

$$p(B) = \lambda {\rm e}^{-\lambda B}, \quad B \ge 0 \tag{457}$$

What is the probability that the total benefit exceeds the critical value $B_{\rm c}$?

(f) Continuing part 94e, suppose that you require that the probability of exceeding the critical benefit, $B_{\rm c}$, must be no less than the critical probability $P_{\rm c}$. However, the critical benefit,

$B_{\rm c}$, is uncertain (you don't really know what you need). Use the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ B_{\rm c} : \left| \frac{B_{\rm c} - \widetilde{B}_{\rm c}}{\widetilde{B}_{\rm c}} \right| \le h \right\}, \quad h \ge 0 \tag{458}$$

Derive an explicit algebraic expression for the robustness function for satisfying the probabilistic requirement.

(g) Repeat part 94a with the following info-gap model:

$$\mathcal{U}(h) = \left\{ b : \left(b - \widetilde{b} \right)^T W^{-1} \left(b - \widetilde{b} \right) \le h^2 \right\}, \quad h \ge 0 \tag{459}$$

where W is a real, symmetric positive definite matrix. W and $\widetilde{b}$ are known. Derive an explicit algebraic expression for the robustness function.

Solution for problem 94: Allocation of scarce resource (p.112).

Part 94a. The robustness function is defined as

$$\widehat{h}(r, B_{\rm c}) = \max \left\{ h : \left(\min_{b \in \mathcal{U}(h)} B(b) \right) \geq B_{\rm c} \right\} \tag{2054}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. Because the allocations, r_n, are non-negative, this inner minimum occurs when each benefit is as small as possible at horizon of uncertainty h:

$$m(h) = \sum_{n=1}^{N} r_n(\widetilde{b}_n - s_n h) = r^T \widetilde{b} - h r^T s \geq B_{\rm c} \quad \Longrightarrow \quad \widehat{h}(r, B_{\rm c}) = \frac{r^T \widetilde{b} - B_{\rm c}}{r^T s} \tag{2055}$$

or zero if this is negative. Note zeroing and trade off.

Part 94b. Eq.(454) implies that allocation r is purportedly better than allocation ρ. To understand the decision-maker's dilemma we can re-write eq.(455) as:

$$\frac{r^T \widetilde{b}}{\rho^T \widetilde{b}} < \frac{r^T s}{\rho^T s} \tag{2056}$$

The left hand side is the ratio of putative benefits, and it exceeds unity: r is putatively better than ρ. However, the righthand side is the ratio of weighted errors, and here we see that r is more uncertain than ρ. The decision maker's dilemma is that r is putatively better but more uncertain than ρ.

Considering the robustness curves in fig. 127 based on eq.(2055), together with the conditions in eqs(454) and (455) on p.112, we see that the robustness curves cross in the positive quadrant. It is clear that allocation r is robust-preferred over allocation ρ for all values of $B_{\rm c}$ in the interval:

$$B_\times < B_{\rm c} < r^T \widetilde{b} \tag{2057}$$

The value of $B_\times$ is obtained by solving:

$$\widehat{h}(r, B_\times) = \widehat{h}(\rho, B_\times) \iff \frac{r^T \widetilde{b} - B_\times}{r^T s} = \frac{\rho^T \widetilde{b} - B_\times}{\rho^T s} \tag{2058}$$

$$\iff B_\times = \left(\frac{1}{\rho^T s} - \frac{1}{r^T s} \right)^{-1} \left(\frac{\rho^T b}{\rho^T s} - \frac{r^T b}{r^T s} \right) \tag{2059}$$

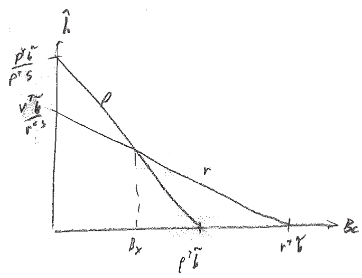

Figure 127: Robustness curves for solution of problem 94b, based on eq.(2055).

Part 94c. The robustness function for program 1 is known and finite for all values of $B_{\rm c}$. The robustness function for program 2 is:

$$\widehat{h}_2(B_{\rm c}) = \begin{cases} \infty & \text{if } B_{\rm c} \leq B_2 \\ 0 & \text{else} \end{cases} \tag{2060}$$

Both robustness functions are shown schematically in fig. 128, employing the fact that $B_2 < B_1$. Program 1 is robust-preferred over Program 2 for:

$$B_2 < B_{\rm c} < B_1 \tag{2061}$$

Program 2 is robust preferred for:

$$B_{\rm c} \le B_2 \tag{2062}$$

We are indifferent for other values of $B_{\rm c}$.

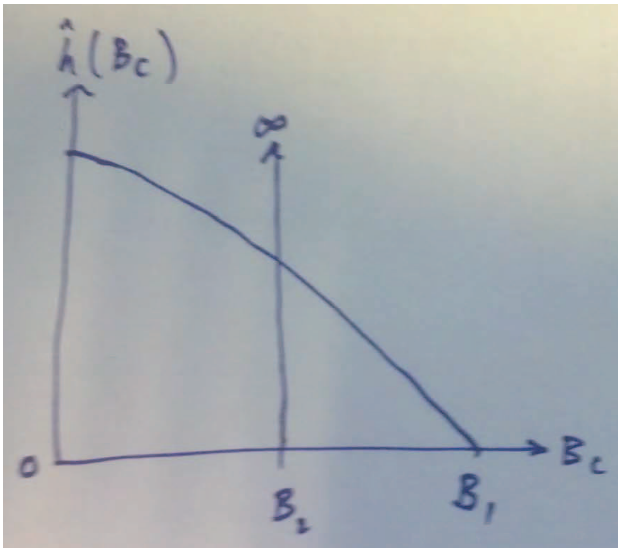

Figure 128: Schematic robustness curves for solution of problem 94c.

Part 94d. The robustness function is defined as in eq.(2054) on p.313, with the new info-gap model. Let $m(h)$ denote the inner minimum, which is obtained by Lagrange optimization. Define the objective function:

$$H = r^T b + \lambda \left[h^2 - (b - \widetilde{b})^T W^{-1} (b - \widetilde{b}) \right] \tag{2063}$$

Extremal values are obtained as follows:

$$\frac{\partial H}{\partial b} = 0 = r - 2\lambda W^{-1}(b - \widetilde{b}) \implies b - \widetilde{b} = \frac{1}{2\lambda} W r \tag{2064}$$

We solve for the Lagrange multiplier by imposing the constraint:

$$(b - \widetilde{b})^T W^{-1} (b - \widetilde{b}) = h^2 \iff \frac{1}{4\lambda^2} (Wr)^T W^{-1} (Wr) = h^2 \iff \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{r^T W r}} \tag{2065}$$

Thus extremal values of H are obtained with:

$$b - \widetilde{b} = \frac{\pm h}{\sqrt{r^T W r}} W r \tag{2066}$$

Thus the inner minimum in the definition of the robustness is:

$$m(h) = r^T \widetilde{b} - \frac{h}{\sqrt{r^T W r}} r^T W r = r^T \widetilde{b} - h\sqrt{r^T W r} \ge B_{\rm c} \tag{2067}$$

Solving for h at equality yields the robustness:

$$\widehat{h}(B_{\rm c}, r) = \frac{r^T \widetilde{b} - B_{\rm c}}{\sqrt{r^T W r}} \tag{2068}$$

or zero if this is negative. Note the structural similarity to eq.(2055): putative value and requirement in the numerator, weighted uncertainty in the denominator.

Part 94e. We integrate the pdf:

$$P(B > B_c) = \int_{B_c}^{\infty} \lambda e^{-\lambda B} \, dB = - e^{-\lambda B} \Big|_{B_c}^{\infty} = e^{-\lambda B_c} \tag{2069}$$

Part 94f. Let $P_s(B_c)$ denote the probability of success, which is $P(B > B_c)$ in eq.(2069). The performance requirement is $P_s(B_c) \ge P_c$ and the robustness function is defined as:

$$\widehat{h}(P_c) = \max \left\{ h : \left(\min_{B_c \in \mathcal{U}(h)} P_s(B_c) \right) \ge P_c \right\} \tag{2070}$$

where the info-gap model, $\mathcal{U}(h)$, is specified in eq.(458) on p.113. Let $m(h)$ denote the inner minimum in eq.(2070). From eq.(2069) we see that $m(h)$ occurs when B_c is maximal at horizon of uncertainty h:

$$m(h) = e^{-\lambda(1+h)\widetilde{B}_c} \ge P_c \iff \lambda(1+h)\widetilde{B}_c \le -\ln P_c \iff \widehat{h}(P_c) = -\frac{\ln P_c}{\lambda \widetilde{B}_c} - 1 \tag{2071}$$

or zero if this is negative. Note zeroing and trade off in fig. 129.

Zeroing: $\widehat{h}(P_c) = 0$ if $P_c = e^{-\lambda \widetilde{B}_c}$ which is the estimated probability of success.

Trade off:

- ◦ $\ln P_c$ approaches $-\infty$ as P_c approaches 0.
- ◦ $\ln P_c$ becomes less negative as P_c increases from 0 to $e^{-\lambda \widetilde{B}_c}$.
- ◦ Hence $\widehat{h}(P_c)$ decreases as P_c increases, as in fig. 129.

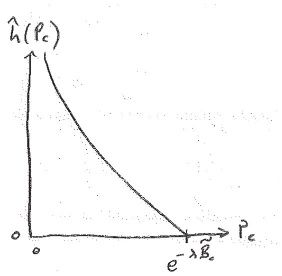

Figure 129: Robustness curve for solution of problem 94f, based on eq.(2071).

Part 94g. The robustness is defined in eq.(2054) with the info-gap model in eq.(459) on p.113. Recall that $B = r^T b$. Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. We derive an expression for $m(h)$ using Lagrange optimization. Define the objective function:

$$H = r^T b + \lambda \left[h^2 - \left(b - \widetilde{b} \right)^T W^{-1} \left(b - \widetilde{b} \right) \right] \tag{2072}$$

Extrema are obtained from:

$$0 = \frac{\partial H}{\partial b} = r - 2\lambda W^{-1} \left(b - \widetilde{b} \right) \implies b - \widetilde{b} = \frac{1}{2\lambda} W r \tag{2073}$$

We solve for λ by using the constraint from the info-gap model:

$$h^2 = \frac{1}{4\lambda^2} r^T W W^{-1} W r \implies \frac{1}{2\lambda} = \pm \frac{h}{\sqrt{r^T W r}} \tag{2074}$$

Thus the total benefit, B, is minimized by:

$$b = \widetilde{b} - \frac{h}{\sqrt{r^T W r}} W r \tag{2075}$$

Finally we obtain:

$$m(h) = r^T\widetilde{b} - \frac{h}{\sqrt{r^TWr}} r^TWr = r^T\widetilde{b} - h\sqrt{r^TWr} \tag{2076}$$

The robustness is the greatest horizon of uncertainty, h, up to which $m(h) \le B_{\rm c}$:

$$\widehat{h}(r, B_{\rm c}) = \frac{r^T\widetilde{b} - B_{\rm c}}{\sqrt{r^TWr}} \tag{2077}$$

or zero if this is negative.