87. **Quantiles with asymmetric uncertainty,** (p.312) $x$ is a non-negative random variable with probability density function (pdf) $p(x)$. The system we are designing will fail if $x$ is too large. We want to know the largest value of $x$ for which the probability of not exceeding this value is $1-\alpha$. This value is called the $(1-\alpha)$ quantile of $x$, denoted $q_\alpha$, and defined in the relation:

$$1-\alpha = \int_0^{q_\alpha} p(x)\,\mathrm{d}x \tag{404}$$

(a) Derive an explicit algebraic expression for the $(1-\alpha)$ quantile of $x$ using the exponential distribution:

$$\widetilde{p}(x) = \widetilde{\lambda}\mathrm{e}^{-\widetilde{\lambda}x} \tag{405}$$

(b) Now suppose that the true pdf of $x$, denoted $p(x)$, is exponential but the coefficient of the distribution, $\lambda$, is uncertain. The best available estimate is $\widetilde{\lambda}$ (which is positive) but we suspect that this is an under estimate. We represent the uncertainty in the pdf of $x$ with this info-gap model:

$$\mathcal{U}(h) = \left\{ p(x) = \lambda\mathrm{e}^{-\lambda x} :\ 0 \le \frac{\lambda-\widetilde{\lambda}}{s} \le h \right\},\quad h \ge 0 \tag{406}$$

where $s$ is a known positive constant. We will estimate the $(1-\alpha)$ quantile using $\widetilde{p}(x)$ in eq.(405), but this will be an over estimate (explain why):

$$0 \le q_\alpha(p) \le q_\alpha(\widetilde{p}) \tag{407}$$

We require that this over estimate not err by more than $\varepsilon$:

$$q_\alpha(\widetilde{p}) - q_\alpha(p) \le \varepsilon \tag{408}$$

Derive an explicit algebraic expression for the robustness if we estimate the quantile as $q_\alpha(\widetilde{p})$.

(c) We continue with the info-gap model of eq.(406) but we estimate the quantile with an exponential distribution whose coefficient, $\lambda_{\mathrm{e}}$, is greater than $\widetilde{\lambda}$. For convenience we will denote quantiles according to the exponential coefficient, so our estimate of the quantile is $q_\alpha(\lambda_{\mathrm{e}})$ and we require that the absolute error of this estimate not exceed $\varepsilon$:

$$|q_\alpha(\lambda_{\mathrm{e}}) - q_\alpha(\lambda)| \le \varepsilon \tag{409}$$

Derive an algebraic expression for the inverse of the robustness function. Explore the crossing of these robustness curves with the robustness curve of part 87b.

**Solution for problem 87: Quantiles with asymmetric uncertainty** (p.101).

**(87a)** The $(1-\alpha)$ quantile with the exponential distribution is defined as:

$$1-\alpha = \int_0^{q_\alpha} \widetilde{\lambda} \mathrm{e}^{-\widetilde{\lambda} x}\, \mathrm{d}x = 1 - \mathrm{e}^{-\widetilde{\lambda} q_\alpha} \implies \alpha = \mathrm{e}^{-\widetilde{\lambda} q_\alpha} \implies \boxed{q_\alpha = -\frac{1}{\widetilde{\lambda}} \ln \alpha} \qquad (1992)$$

**(87b)** The robustness is defined as:

$$\widehat{h}(\varepsilon) = \max \left\{ h :\ \left( \max_{p \in \mathcal{U}(h)} (q_\alpha(\widetilde{p}) - q_\alpha(p)) \right) \le \varepsilon \right\} \qquad (1993)$$

where we recall from eq.(407) on p. 101 that $0 \le q_\alpha(p) \le q_\alpha(\widetilde{p})$. Thus the robustness is infinite if $\varepsilon \ge q_\alpha(\widetilde{p})$. Thus we need only consider $\varepsilon < q_\alpha(\widetilde{p})$.

Let $m(h)$ denote the inner maximum in eq.(1993) which, according to eq.(1992), occurs for $\lambda = \widetilde{\lambda} + sh$ because $p(x)$ is an exponential pdf. Thus:

$$m(h) = q_\alpha(\widetilde{p}) - \frac{-1}{\widetilde{\lambda} + sh} \ln \alpha \le \varepsilon \implies \frac{1}{\widetilde{\lambda} + sh} \ln \alpha \le \varepsilon - q_\alpha(\widetilde{p}) \implies \widetilde{\lambda} + sh \le \frac{\ln \alpha}{\varepsilon - q_\alpha(\widetilde{p})} \qquad (1994)$$

Solving for $h$ at equality we find the robustness:

$$\boxed{\widehat{h}(\varepsilon) = \frac{1}{s} \left( \frac{\ln \alpha}{\varepsilon - q_\alpha(\widetilde{p})} - \widetilde{\lambda} \right)} = \frac{\varepsilon \widetilde{\lambda}^2}{\left( -\ln \alpha - \varepsilon \widetilde{\lambda} \right) s} \qquad (1995)$$

or zero if this is negative. The robustness vanishes at the putative error, which is $\varepsilon = q_\alpha(\widetilde{p}) - q_\alpha(\widetilde{p}) = 0$:

$$\widehat{h}(0) = \frac{1}{s} \left( \frac{\ln \alpha}{-q_\alpha(\widetilde{p})} - \widetilde{\lambda} \right) = \frac{1}{s} \left( \frac{\ln \alpha}{-(-\ln \alpha)/\widetilde{\lambda}} - \widetilde{\lambda} \right) = 0 \qquad (1996)$$

We also see from eq.(1995) that:

$$\lim_{\varepsilon \to q_\alpha(\widetilde{p})} \widehat{h}(\varepsilon) = \infty \qquad (1997)$$

We already know that the robustness is infinite if $\varepsilon \ge q_\alpha(\widetilde{p})$.

**(87c)** The robustness is defined as:

$$\widehat{h}(\varepsilon) = \max \left\{ h :\ \left( \max_{\lambda \in \mathcal{U}(h)} |q_\alpha(\lambda_{\mathrm{e}}) - q_\alpha(\lambda)| \right) \le \varepsilon \right\} \qquad (1998)$$

Let $m(h)$ denote the inner maximum. We see from eq.(1992) that the quantile decreases monotonically as the exponent increases:

$$\frac{\partial q_\alpha(\lambda)}{\partial \lambda} < 0 \qquad (1999)$$

Recall that $\lambda_{\mathrm{e}} \ge \widetilde{\lambda}$. From this we see, from the info-gap model of eq.(406) on p. 101, that $m(h)$ occurs either for $\lambda = \widetilde{\lambda}$ or for $\lambda = \widetilde{\lambda} + sh$. Denote the corresponding values by:

$$\begin{aligned}
m_1 &= \left| q_\alpha(\lambda_{\mathrm{e}}) - q_\alpha(\widetilde{\lambda}) \right| && (2000)\\
&= - \left| \frac{1}{\lambda_{\mathrm{e}}} - \frac{1}{\widetilde{\lambda}} \right| \ln \alpha && (2001)\\
&= - \left( \frac{1}{\widetilde{\lambda}} - \frac{1}{\lambda_{\mathrm{e}}} \right) \ln \alpha && (2002)\\
m_2(h) &= \left| q_\alpha(\lambda_{\mathrm{e}}) - q_\alpha(\widetilde{\lambda} + sh) \right| && (2003)\\
&= - \left| \frac{1}{\lambda_{\mathrm{e}}} - \frac{1}{\widetilde{\lambda} + sh} \right| \ln \alpha && (2004)
\end{aligned}$$

The inner maximum in the robustness is the greater of these two values:

$$m(h) = \max[m_1,\, m_2(h)] \tag{2005}$$