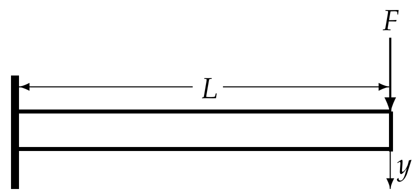

Figure 9: Cantilever for problem 53.

53. **Cantilever–2.** (p.231) Consider the cantilever in fig. 9. The force F is applied perpendicular to the elastic beam of length L which is rigidly constrained at the base. The bending stiffness of the beam is EI and the end deflection is $y = FL^3/(3EI)$.

(a) The anticipated force is $\widetilde{F}$, which is positive. The uncertainty in the true force, F, is represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ F : \left| \frac{F - \widetilde{F}}{\sigma} \right| \leq h \right\}, \quad h \geq 0 \tag{172}$$

where σ is known and positive. The performance requirement is that the end deflection be no less than the critical value y_c. Derive an explicit expression for the robustness to uncertainty.

(b) Continue part (a) and compare two designs with different bending stiffnesses and load uncertainties:

$$(EI)_1 > (EI)_2 \quad \text{and} \quad \sigma_1 < \sigma_2 \tag{173}$$

For what values of critical deflection, y_c, is design $[(EI)_1, \sigma_1]$ preferred over design $[(EI)_2, \sigma_2]$?

(c) Now consider a different performance requirement: the bending moment at the base of the beam must not exceed the critical value M_c. Use the info-gap model of eq.(172) to derive an explicit expression for the robustness to uncertainty.

(d) Derive an expression, based on parts (a) and (c), for the robustness to uncertainty when both of the performance requirements must be satisfied.

(e) Let F be a non-negative random variable with probability density function (pdf) $p(F)$ whose estimated form is exponential: $\widetilde{p}(F) = \lambda \mathrm{e}^{-\lambda F}$. The uncertainty in the pdf is represented by:

$$\mathcal{U}(h) = \left\{ p(F) : \ p(F) \geq 0, \ \int_0^\infty p(F) \, \mathrm{d}F = 1, \ |p(F) - \widetilde{p}(F)| \leq h\widetilde{p}(F) \right\}, \quad h \geq 0 \tag{174}$$

The mechanical system fails if the deflection, y, is less than y_c. The performance requirement is that the probability of failure must not exceed P_c. Derive an explicit expression for the robustness of this performance function, for P_c much less than 1.

(f) Now suppose that N forces, $f = (f_1, \ldots, f_N)$, are applied perpendicularly to the beam, where f_i is applied at a distance ℓ_i from the base. As in part (c), the performance requirement is that the bending moment at the base of the beam must not exceed the critical value M_c. The nominal force vector is $\widetilde{f}$, and uncertainty is represented as:

$$\mathcal{U}(h) = \left\{ f : \ (f - \widetilde{f})^T W (f - \widetilde{f}) \leq h^2 \right\}, \quad h \geq 0 \tag{175}$$

where W is a known, positive definite, symmetric matrix. Derive an explicit expression for the robustness.

(g) Return to part (a) and denote the robustness $\widehat{h}_y$. Suppose that the horizon of uncertainty, h, is a random variable with exponential distribution: $p(h) = \lambda e^{-\lambda h}$. The system fails if the end deflection is less than y_c. Derive an upper bound for the probability of failure, as a function of $\widehat{h}_y$. This upper bound is less than one.

(h) Consider the end-loaded beam in fig. 9, where $L = 1$m and $F = 1000$N. The end deflection was measured 5 times with normal noise, and the observed deflections are 0.016, 0.010, 0.013, 0.011 and 0.012m. Use a statistical test to decide between the following two hypotheses:

$$H_0 : \qquad EI = 2 \times 10^4 \text{Nm}^2 \tag{176}$$

$$H_1 : \qquad EI > 2 \times 10^4 \text{Nm}^2 \tag{177}$$

Do you reject H_0 at 0.05 level of significance?

(i) The beam in fig. 9 is loaded repeatedly and the deflection is measured and categorized as "low", "medium" or "high". Under normal conditions the probabilities of these categories are:

$$p_{\text{low}} = 0.35, \ p_{\text{med}} = 0.55, \ p_{\text{high}} = 0.10 \tag{178}$$

In the last batch of loadings the observations are:

$$n_{\text{low}} = 55, \ n_{\text{med}} = 75, \ n_{\text{high}} = 20 \tag{179}$$

The null hypothesis is that the conditions are normal. Do you reject the null hypothesis at 0.05 level of significance?

Solution for problem 53: Cantilever. (p.51)

(a) The definition of the robustness is:

$$\widehat{h}_y = \max \left\{ h : \left(\min_{F \in \mathcal{U}(h)} y \right) \geq y_{\mathrm{c}} \right\} \tag{1348}$$

Let $\mu_y(h)$ denote the inner minimum, which is the inverse of the robustness. This minimum occurs when F is as small as possible at horizon of uncertainty h: $F = \widetilde{F} - \sigma h$. Thus:

$$\mu_y = \frac{(\widetilde{F} - \sigma h)L^3}{3EI} \tag{1349}$$

Equating this to y_{c} and solving for h yields the robustness:

$$\widehat{h}_y = \frac{1}{\sigma} \left(\widetilde{F} - \frac{3EIy_{\mathrm{c}}}{L^3} \right) \tag{1350}$$

or zero if this is negative. This robustness curve is shown schematically in fig. 69.

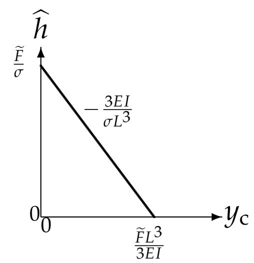

Figure 69: Robustness curve, eq.(1350), problem 53(a), indicating slope and intercepts.

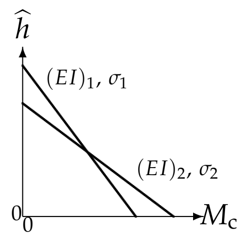

Figure 70: Two robustness curves, eq.(1350), problem 53(b), where $(EI)_1 > (EI)_2$ and $\sigma_1 < \sigma_2$.

(b) The robustness curves, eq.(1350), are shown schematically in fig. 70. They cross when y_{c} equals:

$$y_{\times} = \frac{(\sigma_2 - \sigma_1)\widetilde{F}L^3}{3(EI)_1\sigma_2 - 3(EI)_2\sigma_1} \tag{1351}$$

Design $[(EI)_1, \sigma_1]$ is preferred over design $[(EI)_2, \sigma_2]$ for those values of y_{c} for which design $[(EI)_1, \sigma_1]$ is more robust. Thus design $[(EI)_1, \sigma_1]$ is preferred for $y_{\mathrm{c}} < y_{\times}$.

(c) The bending moment at the base of the beam is $M = FL$. The definition of the robustness is:

$$\widehat{h}_M = \max \left\{ h : \left(\max_{F \in \mathcal{U}(h)} M \right) \leq M_{\mathrm{c}} \right\} \tag{1352}$$

Let $\mu_M(h)$ denote the inner maximum, which is the inverse of the robustness. This minimum occurs when F is as large as possible at horizon of uncertainty h: $F = \widetilde{F} + \sigma h$. Thus:

$$\mu_M = (\widetilde{F} + \sigma h)L \tag{1353}$$

Equating this to M_{c} and solving for h yields the robustness:

$$\widehat{h}_M = \frac{1}{\sigma} \left(\frac{M_{\mathrm{c}}}{L} - \widetilde{F} \right) \tag{1354}$$

or zero if this is negative.

(d) The joint robustness is the lesser of the two marginal robustnesses:

$$\widehat{h} = \min[\widehat{h}_y, \, \widehat{h}_M] \tag{1355}$$

(e) The probability of failure is:

$$P_{\rm f}(p) = {\rm Prob}(y \le y_{\rm c}) = {\rm Prob}\left(\frac{FL^3}{3EI} \le y_{\rm c}\right) = {\rm Prob}\left(F \le \frac{3EIy_{\rm c}}{L^3}\right) \tag{1356}$$

Denote $F_{\rm c} = 3EIy_{\rm c}/L^3$. Thus the probability of failure is:

$$P_{\rm f}(p) = \int_0^{F_{\rm c}} p(F)\,{\rm d}F \tag{1357}$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h : \left(\max_{P\in\mathcal{U}(h)} P_{\rm f}(p)\right) \le P_{\rm c}\right\} \tag{1358}$$

Let $\mu(h)$ denote the inner maximum, which occurs, for very small $P_{\rm c}$, when $p(F)$ is as large as possible for $F \le F_{\rm c}$: $p(F) = (1+h)\widetilde{p}(F)$. Thus:

$$\mu(h) = \int_0^{F_{\rm c}} (1+h)\widetilde{p}(F)\,{\rm d}F = (1+h)[1 - {\rm e}^{-\lambda F_{\rm c}}] \tag{1359}$$

Equating this to $P_{\rm c}$ and solving for h yields the robustness:

$$\widehat{h} = \frac{P_{\rm c}}{1 - {\rm e}^{-\lambda F_{\rm c}}} - 1 \tag{1360}$$

or zero if this is negative.

(f) Denote $\ell = (\ell_1, \ldots, \ell_N)$. The bending moment at the base is $M = \sum_{i=1}^N \ell_i f_i = \ell^T f$. As in eq.(1352), the definition of the robustness is:

$$\widehat{h}_M = \max\left\{h : \left(\max_{f\in\mathcal{U}(h)} M\right) \le M_{\rm c}\right\} \tag{1361}$$

Let $\mu_M(h)$ denote the inner maximum, which we evaluate using Lagrange optimization.

$$H = \ell^T f + \lambda[h^2 - (f-\widetilde{f})^T W (f - \widetilde{f})] \tag{1362}$$

The extremum of H is obtained from:

$$\frac{\partial H}{\partial f} = \ell - 2\lambda W(f-\widetilde{f}) \implies f - \widetilde{f} = \frac{1}{2\lambda} W^{-1}\ell \tag{1363}$$

$$\implies h^2 = \frac{1}{4\lambda^2}\ell^T W^{-1} W W^{-1}\ell \implies \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{\ell^T W^{-1}\ell}} \tag{1364}$$

$$\implies f = \widetilde{f} \pm \frac{h}{\sqrt{\ell^T W^{-1}\ell}} W^{-1}\ell \tag{1365}$$

$$\implies \mu_M(h) = \ell^T f = \ell^T \widetilde{f} + h\sqrt{\ell^T W^{-1}\ell} \tag{1366}$$

Equating this to $M_{\rm c}$ and solving for h yields the robustness:

$$\widehat{h}_M = \frac{M_{\rm c} - \ell^T \widetilde{f}}{\sqrt{\ell^T W^{-1}\ell}} \tag{1367}$$

or zero if this is negative.

(g) We argue as follows:

(i) Failure *cannot* occur if the horizon of uncertainty, h, is no greater than the robustness, $\widehat{h}_y$.

(ii) Failure *can but need not* occur if the horizon of uncertainty, h, exceeds the robustness, $\widehat{h}_y$.

(iii) Hence, the probability of failure is less than the probability that the horizon of uncertainty, h, exceeds the robustness, $\widehat{h}_y$.

Thus:

$$P_{\mathrm{f}} \le \mathrm{Prob}(h > \widehat{h}_y) = \mathrm{e}^{-\lambda \widehat{h}_y} \tag{1368}$$

(h) The sample mean and standard deviation are:

$$\overline{y} = \frac{1}{N}\sum_{i=1}^{N} y_i = 0.0136\mathrm{m} \tag{1369}$$

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{y})^2} = 0.002881\mathrm{m} \tag{1370}$$

If H_0 holds, then:

$$y(H_0) = \frac{FL^3}{3EI} = \frac{1000\mathrm{N} \times 1^3\mathrm{m}^3}{3 \times 2 \times 10^4\mathrm{Nm}^2} = 0.0166666\mathrm{m} \tag{1371}$$

The t statistic, with $N - 1 = 4$ DoF's, is:

$$t = \frac{\overline{y} - y(H_0)}{s/\sqrt{N}} = -2.380 \tag{1372}$$

The level of significance is:

$$\alpha = \mathrm{Prob}(t \le t_{\mathrm{obs}} | H_0) = \mathrm{Prob}(t_{(4)} \le -2.380) < 0.05 \tag{1373}$$

Thus reject H_0 at the 5% level of significance.

(i) The total number of observations is $N = 150$ and the number of categories is $k = 3$. Use the χ^2 statistic, which is:

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - Np_i)^2}{Np_i} \tag{1374}$$

$$= \frac{(55 - 150(0.35))^2}{150(0.35} + \frac{(75 - 150(0.55))^2}{150(0.55)} + \frac{(20 - 150(0.1))^2}{150(0.1)} \tag{1375}$$

$$= 2.4675 \tag{1376}$$

The level of significance for this 2-DoF χ^2 variable is:

$$\alpha = \mathrm{Prob}(\chi^2_{(2)} > 2.4675) > 0.2 \tag{1377}$$

Do not reject H_0 at 0.05 level of significance.