101. **Energy conservation by feedback** (based on exam in 035018, 22.5.2019), (p.350). People change their energy consumption in response to feedback about their prior energy use. Define:

$n(c)\,\mathrm{d}c =$ number of consumers whose prior energy consumption was in the interval $[c,\ c+\mathrm{d}c]$.

The estimated consumption in the next time interval, for a consumer whose prior consumption was in the interval $[c,\ c+\mathrm{d}c]$, is denoted $\widetilde{f}(c,\rho)$, where ρ is a parameter expressing the intensity of the feedback; greater ρ implies greater intensity.

The true consumption function is $f(c,\rho)$, whose uncertainty is represented by an info-gap model, $\mathcal{U}(h)$. The response of the entire population to feedback at intensity ρ is:

$$R(\rho,f) = \int_0^\infty f(c,\rho)n(c)\,\mathrm{d}c \qquad (505)$$

We require that the population response be no greater than the critical value, R_c:

$$R(\rho,f) \le R_\mathrm{c} \qquad (506)$$

(a) Derive an explicit algebraic expression for the robustness function for the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ f(c,\rho):\ f(c,\rho)\ge 0,\ \left| \frac{f(c,\rho)-\widetilde{f}(c,\rho)}{\widetilde{f}(c,\rho)} \right| \le h \right\},\quad h\ge 0 \qquad (507)$$

(b) Derive an explicit algebraic expression for the robustness function for the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ f(c,\rho):\ f(c,\rho)\ge 0,\ \left| \frac{f(c,\rho)-\widetilde{f}(c,\rho)}{w} \right| \le h \right\},\quad h\ge 0 \qquad (508)$$

where w is a known positive constant.

(c) Continuing from part 101b, consider two different situations, $(\rho_1,\ w_1)$ and $(\rho_2,\ w_2)$, where:

$$\rho_2 < \rho_1 \quad \text{and} \quad 0 < w_2 < w_1 \qquad (509)$$

That is, the feedback in situation 1 is more intensive, but the uncertainty in this situation is greater. For what values of R_c is situation 2 robust-preferred? Assume that $\widetilde{f}(c,\rho) = (1-\rho)c$.

(d) For a particular info-gap model, the robustness function takes this form:

$$\widehat{h}(R_\mathrm{c},\rho) = (R_\mathrm{c} - w)\rho \qquad (510)$$

or zero if this is negative, where ρ and w are positive constants. Consider two different situations, $(\rho_1,\ w_1)$ and $(\rho_2,\ w_2)$, where:

$$0 < \rho_2 < \rho_1 \quad \text{and} \quad 0 < w_2 < w_1 \qquad (511)$$

For what values of R_c is situation 1 robust-preferred?

(e) The true and estimated consumption functions are related as:

$$f(c,\rho) = \widetilde{f}(c,\rho) + \sum_{j=1}^{J} a_j \sin\frac{j\pi c}{c_{\max}} \qquad (512)$$

$$= \widetilde{f}(c,\rho) + a^T\sigma(c) \qquad (513)$$

where $c_{\max}$ is a known positive number, and a and $\sigma(c)$ are the vectors of Fourier coefficients and sine functions in eq.(512). The uncertainty in $f(c,\rho)$ is represented by this Fourier-ellipsoid info-gap model:

$$\mathcal{U}(h) = \left\{ f(c,\rho) = \widetilde{f}(c,\rho) + a^T\sigma(c) :\ a^T W a \le h^2 \right\}, \quad h \ge 0 \tag{514}$$

where W is a known, positive definite, real, symmetric matrix. Derive an explicit algebraic expression for the robustness function.

Solution for problem 101. Energy conservation by feedback. (p.123).

101a. The definition of the robustness function is:

$$\widehat{h}(R_c, \rho) = \max\left\{h : \left(\max_{f(c,\rho)\in\mathcal{U}(h)} R(\rho, f)\right) \le R_c\right\} \tag{2270}$$

Let $m(h)$ denote the inner maximum. Because $n(c)$ is non-negative, the inner maximum occurs for $f = (1+h)\widetilde{f}$. Thus:

$$m(h) = \int_0^\infty (1+h)\widetilde{f}(c,\rho)n(c)\,\mathrm{d}c = (1+h)\widetilde{R} \le R_c \tag{2271}$$

where $\widetilde{R}$ is defined implicitly in eq.(2271). Hence:

$$\boxed{\widehat{h}(R_c, \rho) = \frac{R_c}{\widetilde{R}} - 1} \tag{2272}$$

or zero if this is negative.

101b. The inner maximum in the robustness, eq.(2270) is:

$$m(h) = \int_0^\infty \left(\widetilde{f}(c,\rho) + wh\right) n(c)\,\mathrm{d}c = \widetilde{R} + wh\underbrace{\int_0^\infty n(c)\,\mathrm{d}c}_{N} \le R_c \tag{2273}$$

where N is the total number of consumers. Hence:

$$\boxed{\widehat{h}(R_c, \rho) = \frac{R_c - \widetilde{R}}{wN}} \tag{2274}$$

or zero if this is negative.

101c. The robustness curves for the two situations cross at a positive value of R_c, call it $R_\times$. Situation 2 is robust-preferred for R_c values greater than $R_\times$ because $w_2 < w_1$, which implies that situation 2's robustness curve is steeper. We obtain an explicit expression for $R_\times$ as follows. First note that:

$$\widetilde{R} = (1-\rho)\int_0^\infty cn(c)\,\mathrm{d}c = (1-\rho)R_0 \tag{2275}$$

which defines the term R_0. Thus the robustness function in eq.(2274) is:

$$\widehat{h}(R_c, \rho) = \frac{R_c - (1-\rho)R_0}{wN} \tag{2276}$$

Now:

$$\widehat{h}_1(R_\times) = \widehat{h}_2(R_\times) \tag{2277}$$

$$\Longrightarrow \quad \frac{R_\times - (1-\rho_1)R_0}{w_1 N} = \frac{R_\times - (1-\rho_2)R_0}{w_2 N} \tag{2278}$$

$$\Longrightarrow \quad \left(\frac{1}{w_1} - \frac{1}{w_2}\right) R_\times = \left(\frac{1-\rho_1}{w_1} - \frac{1-\rho_2}{w_2}\right) R_0 \tag{2279}$$

$$\Longrightarrow \quad \boxed{R_\times = \frac{(1-\rho_1)w_2 - (1-\rho_2)w_1}{w_2 - w_1} R_0} \tag{2280}$$

We prefer situation 2 for $R_c > R_\times$.

101d. The robustness curves for the two situations cross at a positive value of R_c, call it $R_\times$. Situation 1 is robust-preferred for R_c values greater than $R_\times$ because $\rho_2 < \rho_1$, which implies that situation 1's robustness curve is steeper. We obtain an explicit expression for $R_\times$ as follows.

$$\widehat{h}_1(R_\times) = \widehat{h}_2(R_\times) \tag{2281}$$

$$\Longrightarrow \quad (R_\times - w_1)\rho_1 = (R_\times - w_2)\rho_2 \tag{2282}$$

$$\Longrightarrow \quad (\rho_1 - \rho_2)R_\times = w_1\rho_1 - w_2\rho_2 \tag{2283}$$

$$\Longrightarrow \quad \boxed{R_\times = \frac{w_1\rho_1 - w_2\rho_2}{\rho_1 - \rho_2}} \tag{2284}$$

which is a positive value. We prefer situation 1 for $R_c > R_\times$.

101e. The total population response, $R(\rho, f)$, can be written:

$$R(\rho, f) = \int_0^\infty f(c,\rho)n(c)\,\mathrm{d}c = \int_0^\infty \widetilde{f}(c,\rho)n(c)\,\mathrm{d}c + a^T \underbrace{\int_0^\infty \sigma(c)n(c)\,\mathrm{d}c}_{z} = \widetilde{R} + a^T z \tag{2285}$$

We use Lagrange optimization to find the extreme value for R on the info-gap model at horizon of uncertainty h. Define:

$$H = \widetilde{R} + a^T z + \lambda\left(h^2 - a^T W a\right) \tag{2286}$$

Extrema occur for:

$$0 = \frac{\mathrm{d}H}{\mathrm{d}a} = z - 2\lambda W a \quad \Longrightarrow \quad a = \frac{1}{2\lambda}W^{-1}z \tag{2287}$$

Using the constraint to solve for the Lagrange multiplier:

$$h^2 = \frac{1}{4\lambda^2}z^T W^{-1} W W^{-1} z \quad \Longrightarrow \quad \frac{1}{2\lambda} = \pm\frac{h}{\sqrt{z^T W^{-1} z}} \tag{2288}$$

Thus the inner maximum in the definition of the robustness function is:

$$m(h) = \widetilde{R} + \frac{h}{\sqrt{z^T W^{-1} z}} z^T W^{-1} z = \widetilde{R} + h\sqrt{z^T W^{-1} z} \le R_c \tag{2289}$$

Solving for h at equality yields the robustness function:

$$\boxed{\widehat{h} = \frac{R_c - \widetilde{R}}{\sqrt{z^T W^{-1} z}}} \tag{2290}$$

or zero if this is negative.