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A Non-Probabilistic Concept of Reliability

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Contents

Abstract

Uncertainty can be modelled either probabilistically or non-probabilistically. The former option leads to the concept of reliability as the probability of no-failure. In this paper non-probabilistic convex models of uncertainty are used to formulate reliability in terms of acceptable system performance given uncertain operating environment or uncertain geometrical imperfections. It is shown that probabilistic reliability can be very sensitive to small inaccuracy in the probabilistic model. Consequently, the non-probabilistic concept of reliability is useful when insufficient information is available for verifying a probabilistic model. In addition, a theorem is presented showing that analogous convex and probabilistic models of input uncertainty can lead to very different predictions of the range of output variation.

Keywords: reliability; non-probabilistic; uncertainty modelling; seismic design; shells.

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1 Concepts of Reliability

Reliability has a plain lexical meaning, which the engineers have modified and absorbed into their technical jargon. Lexically, that which is 'reliable' can be depended upon confidently. Applying this to machines or systems, they are 'reliable' (still avoiding technical jargon) if one is confident that they will perform their specified tasks as intended. In current technical jargon, a system is reliable if the probability of failure is acceptably low. This is a legitimate extension of the lexical meaning, since 'failure' would imply behavior beyond the domain of specified tasks. The particular innovation which marks the development of modern engineering reliability is the insight that probability — a mathematical theory — can be utilized to quantify the qualitative lexical concept of reliability.

We do not detract from the importance of the probabilistic concept of reliability by suggesting that probability is not the only starting point for quantifying the intuitive idea of reliability. Probabilistic reliability emphasizes the *probability* of acceptable behavior. Non-probabilistic reliability, as developed here, stresses the *range* of acceptable behavior. Probabilistically, a system is reliable if the probability of unacceptable behavior is sufficiently low. In the non-probabilistic formulation of reliability, a system is reliable if the range of performance fluctuations is acceptably small.

Both approaches grapple with the problem of uncertainty. Both have clear design implications. In both methods, the design variables are viewed as controlling the uncertainty of the performance. In probabilistic reliability, design decisions must reduce the probability of unwanted performance to acceptable levels. In non-probabilistic reliability, the design must assure that the performance remains within an acceptable domain.

2 Models of Uncertainty: A Comparison

Both probabilistic and non-probabilistic concepts of reliability, when construed for purposes of design, attempt to optimize the system with respect to the uncertain factors which influence it. Distinct though overlapping information concerning these uncertainties is required by the two concepts of reliability.

Any probabilistic theory contains two main components: sets of events, and a measure-function defined on these sets.[1] Typically, the sets of events are quite inclusive. For example, the normal distribution extends over the entire real numbers, and probabilities are defined for all subsets. This extravagant gaussian assumption — that anything can occur — is tempered by the probability density function (pdf) which expresses the relative frequency of occurrence of different sets of events.[2]

The non-probabilistic concept of uncertainty is also based on sets of events, but no measure-function on these sets is defined. Instead, information about the uncertainties is invested in the structure of the event-sets. We will concentrate on convex-set models of uncertainty, whose structure is usually specified by meager amounts of information. The non-probabilistic, set-theoretic quantification of uncertainty is typically a poor-man's substitute for probability, containing less information than probabilistic models of uncertainty.

Convex models are one class of non-probabilistic models of uncertainty. A convex model is a convex set[3] of functions, where each function is a possible realization of an uncertain phenomenon. Given specific though limited information which characterizes the uncertain events, one can often

[1] From this perspective, fuzzy logic is probabilistic, since it employs sets of events and membership functions defined on these sets, though the axiomatization of fuzzy logic involves an important distinction from conventional probability [6].

[2] "Your bait of falsehood takes this carp of truth" (II,1:69) [26].

[3] A set S is convex if, for all elements f and g in S and all numbers $0 < \alpha < 1$, the quantity $\alpha f + (1-\alpha)g$ also belongs to S.

define a convex model as the set of *all* functions consistent with this information.[4] This will become clear in the examples to follow. Briefly, however, the energy-bound convex models define sets of functions consistent with a given bound on the energy, while the spectral convex model defines the set of all functions consistent with specific spectral information. Envelope-bound convex models delimit the range of variation of uncertain functions to be consistent with given information, and are a generalization, for functions, of the idea of interval-arithmetic for uncertain parameter values.

The procedure by which one formulates a convex model is basically different from the usual method for specifying a stochastic model. In stochastic formulations one often chooses the form of the model, e.g. gaussian, and then determines the coefficients of that model (mean and covariance in the gaussian case). This procedure can work quite well when the *form* of the model is correct, for then the model-parameters can usually be estimated accurately without the need to sample too extensively. This is because, as in the gaussian model, the parameters can be related to the bulk of events which hover around the mean.

On the other hand, if the *form* of the stochastic model is only approximately correct, then the tails of the calibrated stochastic model may differ substantially from the tails of the actual distribution. This is because the model-parameters, related to low-order moments, are determined from typical rather than rare events. In this case, design decisions will be satisfactory for the bulk of occurrences, but may be less than optimal for rare events. It is the rare events — catastrophes, for example — which are often of greatest concern to the designer. The sub-optimality may be manifested as either an over-conservative or an unsafe design.

We have explained that probabilistic and convex models of uncertainty are structurally different. The former involve probability densities defined on sets of events. The latter involve no measure-functions, but instead determine the structure of the event-sets from the available information about the uncertainty. However, this difference can sometimes be viewed as a matter of degree. In our first example (section 4) we will use a convex model to define a set of allowed probability density functions. An ambient pressure is known to be uncertain, an approximate pdf for this pressure is available, and a convex model describes the set of all possible densities. One could legitimately view this hybrid probabilistic-convex model of uncertainty as simply a collection of probabilistic models. This is more a matter of taste than substance. The crucial point is to recognize the wide range of set-theoretic possibilities for representing uncertainty without specifying probability. Furthermore, seemingly similar probabilistic and convex models of uncertainty can have very different implications for design and reliability, as we will see (section 5). In section 6 we will show that design-for-reliability, in the face of substantial uncertainty, can be pursued without reference to probabilistic ideas at all. In our final example (section 7) we present a reliability analysis for geometrical imperfections. However, before elaborating on the non-probabilistic concept of uncertainty and its applications to reliability, it is appropriate to consider the limitations of probability theory in technology. This we do in section 3.

3 Limitations of Probability

The mathematical theory of probability has proven useful in many technological applications. However, it has limitations which, when clearly identified, facilitate our understanding of the non-probabilistic alternatives.

One criticism of probabilistic concepts of uncertainty arises in discussion of prior probability and Bayesian inference and decision theory. A classical objection to Bayesian statistics hits at the source of the prior distribution and utility functions. As Isaac Levi asserts: "Strict Bayesians are

[4] That sets defined in this manner often turn out to be convex is surprising and significant. A partial explanation of the origin of this convexity is found in section 2.1 of [5].

legitimately challenged to tell us where they get their numbers." [20, p 387]. In outlining "the general statistical decision problem", Fenstad notes that "[T]he difficulty arises in connection with the prior measure ... . Does every set of alternatives carry a probability?" [13, pp 2–3] and if so, what does it mean? Furthermore, uniqueness in formulating prior distributions is illusive: a given quantity of prior information is often not represented by a unique prior probability distribution [28].

Before the twentieth century, it was common to identify ignorance of likelihoods with equality of probabilities. Many of Lewis Carroll's probabilistic riddles are based upon this assumption [7]. Also, it appears in the solution of Buffon's needle-problem, from the 18th century. On Buffon's solution Coolidge has remarked that Buffon failed to recognize "the great dangers involved in assuming the equally likely" [8]. John Venn also used the uniform distribution as the fundamental device for describing lack of information. He recognized, however, that "[a]ny attempt to draw inferences from the assumption of (uniform) random arrangement must postulate the occurrence of this particular state of things at some stage or other. But there is often considerable difficulty, leading occasionally to some arbitrariness, in deciding the particular stage at which it ought to be introduced." [32, p97]

The difficulty of quantifying prior knowledge is seen quite clearly in such quandries as the prisoner's dilemma [17] and similar riddles [14] where alternative decisions each seem fully consistent with the initial information. Considering the criticism of Bayesian priors, together with these riddles whose formulation is sparse and simple yet whose resolution has taxed the attention of many people, one may be inclined to agree with Kyburg that "it might be the case that some novel procedure could be used in a decision theory that is based on some non-probabilistic measure of uncertainty." [19, p 189].

Perhaps such thinking as this led Suppes and Zanotti to stress the "distinction between indeterminacy and uncertainty". Their concepts of upper and lower probabilities "are defined in a purely set-theoretical way and thus do not depend ... on explicit probability considerations." [31, p 427]. They continue:

> For a strict Bayesian there is no indeterminacy, for he would postulate a prior probability ... and thereby obtain a standard random variable ... . The concept of indeterminacy is a concept for those who hold that not all sources of error, lack of certain knowledge, etc., are to be covered by a probability distribution, but may be expressed in other ways, in particular, by random relations as generalizations of random variables, and by the resulting concepts of upper and lower probabilities. [31, p 434].

In a different vein, we must mention the reductionist view, as expressed by De Finetti: "Probabilistic reasoning — always to be understood as subjective — merely stems from our being uncertain about something." Uncertainty is the more primitive concept, while probability is a mathematical construction: "probability does not exist" [10, p x]. Indeed, in Kolmogorov's 1933 axiomatization of probability, this theory is put in its "natural place, among the general notions of modern mathematics", with no more than a passing reference to the "concrete physical problems" from which probability theory arose. [18, p v]. This formalistic attitude might suggest the possibility of other mathematical theories describing the same phenomena yet subject to different, non-probabilistic, interpretation. (De Finetti, however, does not seem to have this in mind [9].)

Let us consider the statisticians themselves. The theories of distribution-free inference and non-parametric statistics [15, 16] are motivated by the need to draw conclusions without assuming specific probabilistic properties for the underlying populations from which data are drawn. One can not impute 'non-probabilistic' tendencies to the proponents of these statistical theories. However, the considerable interest in non-parametric statistics attests to the difficulty one may encounter in implementing, or justifying, those statistical methods which are based on assuming specific prior or conditional probability distributions.

One of the primary driving forces in the origin of non-probabilistic models of uncertainty in the engineering community has been precisely this difficulty. Referring to turbulent wind fluctuations acting on transport aircraft or tall buildings, Sobczyk and Spencer [29, p89] enumerate numerous complicating factors and conclude that "the engineering analysis of fatigue reliability assumes some *standard* representations of the spectrum of a turbulent wind." (Italics in the original). The assumption of standard representations arises from the difficulty of verifying more specific models. Considering steel offshore platforms they assert that "the establishment of standard load spectra ... [is] much more difficult than for aircraft structures."

In a similar vein, Murota and Ikeda develop a theory for buckling of trusses with geometrical imperfections, and comment that they

> have employed random imperfections ... although it is somewhat hypothetical at this stage, since the probability distribution cannot be known precisely in practice. The present analysis is not independent of the hypothetical distribution, and the quantitative aspects of the results will have limitations in applicability. However, the qualitative aspects of the conclusions will remain valid for a wide range of probability distributions. [23].

Design-for-reliability would seem to depend on quantitative results, not only qualitative ones.

Let us briefly consider some non-probabilistic treatments of uncertainty in engineering. Drenick [11, 12] and Shinozuka [27] describe uncertain seismic loads on civil structures by defining sets of possible input functions, with no probability measures defined on these sets. Schweppe [24] and Witsenhausen [33, 34] describe estimation and control algorithms for linear dynamic systems based on sets of inputs. Schweppe [25] develops inference and decision rules based on assuming the uncertain phenomenon can be quantified in such a way as to be bounded by an ellipsoid, again with no probability function involved. Ben-Haim [1] develops a method for optimal design of material assay systems based on convex sets of uncertain spatial distributions of the analyte material. Ben-Haim and Elishakoff [5] describe a range of analysis and design problems in applied mechanics based on defining convex sets of uncertain input functions or uncertain geometrical imperfections. Lindberg [21, 22] and Ben-Haim [2] use the convex modelling method to study radial pulse buckling of geometrically imperfect thin-walled shells.

There seem to be numerous suggestions, both among philosophers and technologists, that one's thinking about uncertainty can be something other than probabilistic. We hope in this article to loosen the link between uncertainty and probability, and to suggest a specific non-probabilistic methodology for reliability analysis.

4 Sensitivity of the Failure Probability: An Example

Let us consider failure by rupture of a long cylindrical tube subject to uncertain internal pressure, P. From the perspective of probabilistic reliability, we wish to choose the wall thickness to assure that the probabilility of failure is no larger than a specified value. However, we do not precisely know the probability density function (pdf) of the pressure. We will see that small errors in the pdf can lead to large errors in the probability of failure. A similar example, with a different formulation of the uncertainty, is discussed in [5, pp11–13].

4.1 Uncertainty in the PDF of the Load

The equivalent stress for Tresca's maximum shear stress failure criterion is:

$$\sigma_{\text{eq}} = P/\rho, \quad \text{where} \quad \rho = \left(r_2^2 - r_1^2\right) \Big/ 2r_2^2 \tag{1}$$

where r_1 and r_2 are the inner and outer radii, respectively. Failure occurs if σ_{eq} exceeds the yield stress, σ_{y}. If the wall thickness, h, is very small, $h = r_2 - r_1 \ll r_1$, then $\rho \approx h/r_1$.

We suppose that the pdf of the pressure is in fact a complicated and imprecisely known function. For design purposes, however, we approximate the pdf as an exponential density:

$$f_0(p) = \beta_0 e^{-\beta_0 p}, \quad p \geq 0 \tag{2}$$

In fact the real pdf is:

$$f_\eta(p) = [\alpha + \eta(p)]\, e^{-\beta_0 p}, \quad p \geq 0 \tag{3}$$

where $\eta(p)$ is an unknown function, and α is a constant which normalizes the pdf:

$$\alpha = \beta_0 \left[1 - \int_0^\infty \eta(p) e^{-\beta_0 p}\, dp \right] \tag{4}$$

If $\eta(p)$ is constant, then f_0 and f_η are identical. If $|\eta(p)| \ll \beta_0$, then $f_0(p)$ would seem to be a good approximation to $f_\eta(p)$.

We will use an envelope-bound convex model [5] to represent the allowed range of variation of the functions $\eta(p)$. The set of possible η-functions is:

$$\mathcal{F}_{\text{ENV}} = \{\eta(p) : \ \eta_1(p) \leq \eta(p) \leq \eta_2(p)\} \tag{5}$$

where $\eta_1(p)$ and $\eta_2(p)$ are known non-negative functions which envelop the range of variation of the perturbation, $\eta(p)$.[5]

The mean of p on $f_0(p)$ is $1/\beta_0$. Employing eq.(4) one finds the mean of p on f_η is:

$$\text{E}_{f_\eta}(p) = \frac{1}{\beta_0} + \int_0^\infty \left(p - \frac{1}{\beta_0} \right) \eta(p) e^{-\beta_0 p}\, dp \tag{6}$$

If the perturbation, $\eta(p)$, is small, or occurs far out on the tail of the exponential, then the means of f_0 and f_η are nearly equal.

The maximum mean, for $\eta \in \mathcal{F}_{\text{ENV}}$, occurs when $\eta(p)$ switches from the lower to the upper envelope when the rest of the integrand of eq.(6) changes sign from negative to positive:

$$\max_{\eta \in \mathcal{F}_{\text{ENV}}} \text{E}_{f_\eta}(p) = \frac{1}{\beta_0} + \int_0^{1/\beta_0} \left(p - \frac{1}{\beta_0} \right) \eta_1(p) e^{-\beta_0 p}\, dp + \int_{1/\beta_0}^\infty \left(p - \frac{1}{\beta_0} \right) \eta_2(p) e^{-\beta_0 p}\, dp \tag{7}$$

Let us choose the following envelope functions, for which one can verify that the densities $f_\eta(p)$ are non-negative and normalizable by α for all η-functions if ν is sufficiently small.

$$\eta_1(p) = 0 \tag{8}$$

$$\eta_2(p) = \begin{cases} 0 & p < p_1 \\ \nu e^{-\gamma(p - p_1)} & p \geq p_1 \end{cases} \tag{9}$$

Assuming $p_1 \geq 1/\beta_0$, one finds the maximum expectation on f_η, for $\eta \in \mathcal{F}_{\text{ENV}}$, to be:

$$\max_{\eta \in \mathcal{F}_{\text{ENV}}} \text{E}_{f_\eta}(p) = \frac{1}{\beta_0} \left[1 + \nu \frac{p_1 \beta_0 (\gamma + \beta_0) - \gamma}{(\gamma + \beta_0)^2} e^{-\beta_0 p_1} \right] \tag{10}$$

For example, let $\beta_0 = \gamma = 1$, $p_1 = 4$ and $\nu = 10^{-3}$. Then $\text{E}_{f_0}(p) = 1$, while $\max_\eta \text{E}_{f_\eta}(p) - \text{E}_{f_0}(p) \approx 3.2 \times 10^{-5}$. The nominal density is determined by a single parameter, β_0. In this example it would be quite difficult to distinguish between f_0 and f_η, for any $\eta \in \mathcal{F}_{\text{ENV}}$. Yet we will see that the failure probabilities and design decisions can be quite different for these different probability densities.

[5] Care must be taken in the choice of η_1 and η_2 to assure that each f_η is always non-negative and normalizable by α.

4.2 Sensitivity of the Failure Probability

The probability of failure by rupture equals the probability that the pressure will rise to such a level that the equivalent stress will exceed the yield stress. For pdf f_η, the probability of failure is:

$$\varphi_\eta = \mathrm{Prob}\,(\sigma_{\mathrm{eq}} \geq \sigma_{\mathrm{y}}) = \mathrm{Prob}\,(P \geq \sigma_{\mathrm{y}}\rho) = \int_{\sigma_{\mathrm{y}}\rho}^{\infty} f_\eta(p)\, dp \qquad (11)$$

$$= e^{-\beta_0\sigma_{\mathrm{y}}\rho} + \int_0^\infty \left[H(p - \sigma_{\mathrm{y}}\rho) - e^{-\beta_0\sigma_{\mathrm{y}}\rho} \right] \eta(p) e^{-\beta_0 p}\, dp \qquad (12)$$

where α from eq.(4) has been substituted into eq.(3), and where $H(x) = 1$ if $x \geq 0$, and $H(x) = 0$ otherwise. The first term in eq.(12) is the probability of failure based on the nominal pdf of the pressure, f_0; the second term expresses the contribution of the uncertainty in $f_\eta(p)$.

It is an elementary matter to evaluate the greatest probability of failure, for any η-function in $\mathcal{F}_{\mathrm{ENV}}$. The maximum of φ_η in eq.(12) occurs when $\eta(p)$ switches from its lower to its upper envelope as the term in square brackets changes in sign from negative to positive. Assume that $\sigma_{\mathrm{y}}\rho \leq p_1$. One finds the maximum probability of failure, with the envelopes of eqs.(8) and (9), to be:

$$\widehat{\varphi} = \max_{\eta \in \mathcal{F}_{\mathrm{ENV}}} \varphi_\eta \qquad (13)$$

$$= e^{-\beta_0\sigma_{\mathrm{y}}\rho} + \frac{\nu \left(1 - e^{-\beta_0\sigma_{\mathrm{y}}\rho}\right) e^{-\beta_0 p_1}}{\beta_0 + \gamma} \qquad (14)$$

Let us consider the thin-walled case, so $\rho \approx h/r_1$. From the strict probabilistic point of view, it is reasonable to choose the tube-wall thickness, h, on the basis of the available probabilistic information, $f_0(p)$. One chooses h to achieve a specified probability of failure, φ_0, from eq.(12) with $\eta = 0$:

$$\varphi_0 = e^{-\beta_0\sigma_{\mathrm{y}} h/r_1} \qquad (15)$$

Substituting this nominal design decision into eq.(14) for the maximum probability of failure with the actual pdf yields:

$$\widehat{\varphi} = \varphi_0 + \frac{\nu\,(1 - \varphi_0)\, e^{-\beta_0 p_1}}{\beta_0 + \gamma} \qquad (16)$$

The actual probability of failure, $\widehat{\varphi}$, can be substantially greater than the value, φ_0, upon which the wall-thickness is chosen. For example, choose $1 = \beta_0 = \gamma$ and $\nu = 10^{-3}$ as before. If the desired probability of failure is $\varphi_0 = 10^{-6}$ and the disturbance in $f(p)$ appears at four standard deviations from the origin ($p_1 = 4$), then eq.(16) indicates that $\widehat{\varphi}/\varphi_0 = 10.2$. That is, the actual probability of error is 10 times the design value. $\widehat{\varphi}$ is of course still a small number, but not as small as φ_0.

4.3 Design Implications

Let us continue with the thin-walled case. In the ordinary probabilistic analysis the wall-thickness is chosen by inverting eq.(15) as:

$$h_{\mathrm{p}} = -\frac{r_1}{\beta_0\sigma_{\mathrm{y}}} \ln \varphi_0 \qquad (17)$$

If we include the non-probabilistic information about the uncertainty in the pdf, namely the convex model $\mathcal{F}_{\mathrm{ENV}}$, then the wall thickness is chosen by equating $\widehat{\varphi}$ to φ_0 and inverting eq.(14):

$$h_{\mathrm{cm}} = -\frac{r_1}{\beta_0\sigma_{\mathrm{y}}} \ln \frac{\varphi_0 - \zeta}{1 - \zeta} \qquad (18)$$

where $\zeta = \nu \exp(-\beta_0 p_1)/(\beta_0 + \gamma)$. (Note that eq.(14) cannot be solved for h unless $\zeta < \varphi_0$. However, this is presumably an artifact of the thin-wall restriction.)

It is now possible to compare the strict probabilistic design, $h_{\rm p}$, with the probabilistic design which has been augmented by a convex model for uncertainty in the pdf of the pressure, $h_{\rm cm}$. For example, suppose $1 = \beta_0 = \gamma$, $\nu = 10^{-3}$ and $p_1 = 4$ as before. Then $\zeta = 9.158 \times 10^{-6}$, and (augmented) thin-walled designs are available at any reliability $\varphi_0 > \zeta$. For example, a probability of failure of $\varphi_0 = 10^{-5}$ results in a ratio of the design thicknesses of $h_{\rm cm}/h_{\rm p} = 1.2$; the ordinary probabilistic design is 20% too thin.

The strict probabilistic design is under-conservative in this example because the actual pdf functions, $f_\eta(p)$, are all biased (very slightly) towards higher pressures than the nominal pdf, $f_0(p)$. If the functions $\eta(p)$ were slightly negative rather than slightly positive the reverse situation would probably arise: the strict probabilistic design would be overly conservative.

The point of this example is that very small uncertainties in the pdf, located far from the bulk of events, are difficult to detect but cause substantial inaccuracy in both design-decisions and assessment of failure-probability.

5 A Stochastic Comparison

One main point of this paper is the use of convex models of uncertainty for assessing performance-sensitivity to variation in input or structural factors. The ability to use convex models in this way is fundamental to the non-probabilistic concept of reliability. In this section we discuss a result showing that analogous stochastic and convex models of uncertainty can lead to very different predictions of the range of output variation. This has important consequences for the evaluation of reliability.

We will compare the integral energy-bound convex model against an analogous stochastic model and show that the latter predicts substantially smaller projected responses.

Many types of failures in dynamical mechanical systems are associated with unduly large fluctuations in variables such as displacement or acceleration. In design-for-reliability of such systems one attempts to choose the design parameters to minimize the large oscillations. We will demonstrate that analogous stochastic and convex models of uncertainty can lead to very different predictions of the range of output variation. Consequently, both the design decisions and the reliability will differ substantially when based on these two different uncertainty models.

Consider a linear dynamic system in state-space representation:

$$\dot{x}(t) = Ax(t) + Bf(t) \tag{19}$$

where A and B are constant matrices, $x(t) \in \mathcal{R}^N$ is the state vector and $f(t) \in \mathcal{R}^{N_I}$ is an uncertain input.

In the integral energy-bound convex model the integral over time of the energy of the disturbance is bounded. The set of allowed input vector functions is:

$$\mathcal{F}_{\rm IEB} = \left\{ f(t) : \ \int_0^t f^T(\tau) f(\tau)\, d\tau \le \rho_I^2(t) \right\} \tag{20}$$

where the superscript T implies matrix transposition.

For zero initial conditions, the solution of eq.(19) can be projected along a constant direction-vector ψ and expressed as:

$$\psi^T x(t) = \int_0^t \psi^T {\rm e}^{A(t-\tau)} B f(\tau)\, d\tau \tag{21}$$

The projection direction, ψ, is chosen to emphasize relevant dynamics. For example, ψ can be a modal direction, or ψ can be chosen so that the projection is the difference between displacement at two nodes of the system. Projected responses are scalars and linear functions of f, so they are much easier to optimize than, for example, the norm of x.

Using the Cauchy and the Cauchy-Schwarz inequalities, one can derive the following expression for the maximum projection along ψ allowed by $\mathcal{F}_{\rm IEB}$:

$$\pi_\psi^{IEB}(t) = \max_{f\in\mathcal{F}_{\rm IEB}} \psi^T x(t) \tag{22}$$

$$= \rho_I(t)\sqrt{\int_0^t \psi^T {\rm e}^{A\tau} B B^T {\rm e}^{A^T\tau}\psi \, d\tau} \tag{23}$$

We wish to compare this maximum-response prediction of the integral energy-bound convex model against an analogous stochastic model, to show that even when the input-uncertainty models seem quite similar, their output predictions are quite different. Stochastic homologies for several convex models are discussed in [3].

In the stochastic model of the uncertainty, let the input vector, $f(t)$, be a zero-mean random process with covariance matrix:

$$\mathrm{E}\left[f(t)f^T(\tau)\right] = \begin{cases} \delta(t-\tau)\sigma_f^2 I & t \neq \tau \\ \sigma_f^2 I & t = \tau \end{cases} \tag{24}$$

where σ_f is a scalar and I is the identity matrix.

We wish to choose the convex model to be "equivalent" to the stochastic model, in some reasonable sense. Let us choose ρ_I^2 as the statistical expectation of the integrated disturbance energy. Thus $\mathcal{F}_{\rm IEB}$ contains all input vectors whose integrated energy is bounded by the average integrated energy of the stochastic model. ρ_I^2 is:

$$\rho_I^2(t) = \mathrm{E}\left[\int_0^t f^T(\tau)f(\tau)\, d\tau\right] = N_I \sigma_f^2 t \tag{25}$$

The result on the righthand side is obtained by manipulations with the trace operator.

In the random process, $f^T(t)f(t)$ can of course attain values in excess of its mean value. In other words, this choice of ρ_I^2 defines a rather restrictive convex model, since it very strictly constrains the input vectors. It is, however, a convex model which is homologous to the stochastic inputs at one standard deviation of the energy. In other words, this value of ρ_I^2 defines a convex model whose elements are quite similar to stochastic inputs up to one standard deviation. To define a convex model which is homologous to the stochastic process at, for instance, α standard deviations, one would choose ρ_I^2 as α^2 times the value in eq.(25). Our aim is simply to compare the stochastic and convex model predictions of the range of response variation, so the scale factor α is irrelevant.

We will compare π_ψ^{IEB}, based on $\mathcal{F}_{\rm IEB}$ with ρ_I^2 from eq.(25), against the stochastic value of $\psi^T x$ at one standard deviation, from zero initial conditions. The mean of $\psi^T x$ vanishes and its standard deviation is:

$$\sigma_{\rm proj}(t) = \sigma_f \sqrt{\int_0^t \psi^T {\rm e}^{A\tau} B B^T {\rm e}^{A^T\tau}\psi \, d\tau} \tag{26}$$

Now, employing eqs.(23) and (25), we conclude that:

$$\pi_\psi^{IEB}(t) = \sigma_{\rm proj}(t)\sqrt{N_I t} \tag{27}$$

This relation asserts that the maximum response at any time $t > 1/N_I$, predicted by a convex model which is 'calibrated' to include inputs up to one standard deviation of the integrated energy, will exceed the stochastic response at one standard deviation. Furthermore, this excess of π_ψ^{IEB} over $\sigma_{\rm proj}$ increases monotonically in time. The same conclusion will recur at whatever number, α, of standard deviations one examines, since α would just multiply each side of eq.(27). For $t < 1/N_I$, the convex-model bound converges to zero, as $t \to 0$, more rapidly than the probabilistic standard deviation.

One typical design-for-safety strategy is to choose the design-parameter values so as to achieve acceptably small outputs. The stochastic outputs, being less than the analogous convex model predictions for $t > 1/N_I$ as shown by eq.(27), will lead the designer to choose different parameters in the stochastic-model than in the convex-model analysis. In other words, the stochastic design will tend to allow greater responses than the convex-model design operating in the same conditions.

This conclusion can be stated differently, by noting that the convex model $\mathcal{F}_{\text{IEB}}$ with ρ_I^2 chosen from eq.(25), contains *all* input functions $f(t)$ consistent with the constraint $\int_0^t f^T(\tau)f(\tau)\,d\tau \le \text{E}\left[\int_0^t f^T(\tau)f(\tau)\,d\tau\right]$. Qualitatively speaking, these input functions "look" like white stochastic inputs up to one standard deviation of the energy. However, the stochastic outputs at one standard deviation "overlook", as it were, some of the input functions allowed by the convex model.

Further insight into eq.(27) is obtained by inverting the interpretation of the integral energy-bound convex model formulated with ρ_I^2 from eq.(25). Sometimes a stochastic model is formulated by starting from semi-quantitative data about the range of variability of the uncertain function, then choosing the form of the stochastic model and then fitting the parameters of the model to the data. In the present case, the initial data might correspond to the assertions that $f(t)$ varies around zero and that $\int_0^t f^T(\tau)f(\tau)\,d\tau$ typically does not exceed $\rho_I^2(t)$. The stochastic model adopted is that $f(t)$ is a zero-mean white process for which $\text{E}\left[\int_0^t f^T(\tau)f(\tau)\,d\tau\right] = \rho_I^2(t)$. Eq.(24) is a covariance matrix consistent with this model, when $\sigma_f^2 = \rho_I^2/N_I t$. It results, however, that this stochastic model predicts much lower output responses at one standard deviation than the convex model which includes all input functions consistent with $\int_0^t f^T(\tau)f(\tau)\,d\tau \le \rho_I^2(t)$. Again, increasing the number of standard deviations does not change the conclusion, since the stochastic model at α standard deviations is analogous to the convex model for which $\int_0^t f^T(\tau)f(\tau)\,d\tau \le \alpha^2\rho_I^2(t)$.

6 Non-Probabilistic Reliability: A Seismic Example

Seismically-safe design of buildings is not limited to assuring structural integrity alone. Also important is the functional integrity of critical secondary equipment such as communications units, fire-control facilities, and so on. Building codes for seismically-safe structures require that the designer guarantee specified limits to the inertial forces acting on critical secondary equipment during earthquakes [30].

In this section we will use a convex model to represent uncertainty in the temporal waveform of an earthquake excitation, and derive an expression for the maximum inertial force exerted on a piece of light equipment which is dynamically coupled to a building. The convex model is chosen to include all waveforms consistent with given spectral information about seismic ground motion. The analysis which we will perform enables the designer to choose the dynamical coupling of the equipment to the building so that the inertial forces acting on the equipment during any earthquake represented by the convex model will be within acceptable bounds. The equipment is then 'reliable' in our non-probabilistic sense: no earthquake consistent with available data on ground-motion variability will exert unacceptable forces on the equipment.

A major limitation of this analysis is its dependence on an accurate dynamical model for the motion of the building during an earthquake and for the dynamical coupling of the building to the equipment. This however is characteristic of any model-based dynamical analysis.

An additional limitation is the paucity of information upon which the convex model is founded. However, the adverse effect of limited information about seismic variability cannot properly be viewed as a limitation of the analysis, but rather an inherent deficiency in the informational infrastructure upon which the analysis rests. To the extent that the convex model includes all earthquake ground motions consistent with existing information, the convex model is faithful to the information, without introducing additional strong presumptions about earthquake behavior. In a probabilistic model, the adoption of a specific analytical form for the probability density is usually an assumption whose verification is at best fragmentary for those rare events against which the design must particularly guard.

It must of course be recognized that a convex model does introduce presumptions or extrapolations beyond the raw data. A convex model contains an infinity of functions while the primary observations are surely finite. A central question in the comparison of probabilistic and convex models of uncertainty, and their application to design and reliability, is the evaluation of the relative potency of the presumptions which these models introduce. The result discussed in section 5 is a step in this direction.

6.1 Dynamics

The structure to which the equipment is attached is an N-dimensional linear elastic system with viscous damping:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Bu(t) \tag{28}$$

where M, C and K are constant mass, damping and stiffness matrices, respectively, $x(t)$ is the deflection vector, $u(t)$ is the N_I-dimensional input and B is a constant $N \times N_I$ matrix.

We will assume that C is diagonalizable by the modal vectors. We will also assume that the nth mode is dominant during seismic excitation, so the displacement of the ith node of the structure is:

$$x_i(t) \approx \phi_i^n \eta_n(t) = \frac{\phi_i^n}{m_n \omega_{nD}} \int_0^t \phi^{nT} Bu(\tau) \mathrm{e}^{-\zeta_n \omega_n (t-\tau)} \sin \omega_{nD}(t-\tau)\, d\tau \tag{29}$$

where ϕ^n is the nth mode-shape vector and $\eta_n(t)$ is the nth modal coordinate. m_n, $\omega_n$and ζ_n are the modal mass, undamped natural frequency and the damping ratio for the nth mode. Also define: $\omega_{nD} = \omega_n \sqrt{1-\zeta_n^2}$.

The secondary equipment attached to the ith node has mass, damping ratio and undamped natural frequency m_e, ζ_e and ω_e. Define $c_e = 2\zeta_e \omega_e m_e$, $\omega_{eD} = \omega_e \sqrt{1-\zeta_e^2}$ and $k_e = \omega_e^2 m_e$. We adopt the assumption of 'cascaded dynamics', which asserts that the motion of the equipment is driven by the floor motion, but that the dynamics of the building are uneffected by the equipment motion. The equation of motion for the equipment is:

$$m_e \ddot{y}(t) + c_e [\dot{y}(t) - \dot{x}_i(t)] + k_e [y(t) - x_i(t)] = 0 \tag{30}$$

where $y(t)$ is the equipment displacement with respect to the ground. Define:

$$\alpha(t) = \mathrm{e}^{-\zeta_n \omega_n t} [(k_e - c_e \zeta_n \omega_n) \sin \omega_{nD} t + c_e \omega_{nD} \cos \omega_{nD} t] \tag{31}$$

$$\sigma(t) = \mathrm{e}^{-\zeta_e \omega_e t} \sin \omega_{eD} t, \quad \gamma = \frac{\phi_i^n}{m_e m_n \omega_{eD} \omega_{nD}} \tag{32}$$

With these definitions, the deflection of the equipment becomes:

$$y(t) = \gamma \int_0^t \phi^{nT} Bu(\theta) \underbrace{\int_\theta^t \sigma(t-\tau) \alpha(\tau-\theta)\, d\tau}_{\nu(t,\theta)}\, d\theta \tag{33}$$

which defines the function $\nu(t,\theta)$ which will appear repeatedly in our analysis. After rather arduous computations one finds:

$$\nu(t,\theta) = m_e \omega_e \widetilde{\nu}(t,\theta) \tag{34}$$

where $\widetilde{\nu}(t,\theta)$ is a dimensionless function defined as:

$$\begin{aligned}\widetilde{\nu}(t,\theta) &= \frac{1}{2}\mathrm{e}^{-\zeta_n\omega_n(t-\theta)}\left[(\gamma_1+\gamma_3)\cos\omega_{nD}(t-\theta)+(\gamma_2-\gamma_4)\sin\omega_{nD}(t-\theta)\right] \\ &\quad -\frac{1}{2}\mathrm{e}^{-\zeta_e\omega_e(t-\theta)}\left[-(\gamma_2+\gamma_4)\sin\omega_{eD}(t-\theta)+(\gamma_1+\gamma_3)\cos\omega_{eD}(t-\theta)\right] \end{aligned} \tag{35}$$

The following dimensionless coefficients are employed:

$$\gamma_1 = \frac{\beta_1\beta_3+\beta_2\beta_{4+}}{\beta_{5+}}, \quad \gamma_2 = \frac{\beta_1\beta_{4+}-\beta_2\beta_3}{\beta_{5+}} \tag{36}$$

$$\gamma_3 = \frac{\beta_2\beta_{4-}-\beta_1\beta_3}{\beta_{5-}}, \quad \gamma_4 = -\frac{\beta_1\beta_{4-}+\beta_2\beta_3}{\beta_{5-}} \tag{37}$$

$$\beta_1 = 1-2\zeta_e\zeta_n\frac{\omega_n}{\omega_e}, \quad \beta_2 = 2\frac{\omega_n}{\omega_e}\zeta_e\sqrt{1-\zeta_n^2}, \quad \beta_3 = \zeta_e-\zeta_n\frac{\omega_n}{\omega_e} \tag{38}$$

$$\beta_{4\pm} = \sqrt{1-\zeta_e^2}\pm\frac{\omega_n}{\omega_e}\sqrt{1-\zeta_n^2}, \quad \beta_{5\pm} = \beta_3^2+\beta_{4\pm}^2 \tag{39}$$

The integral $\nu(t,\theta)$ in eq.(33) always exists, but eq.(35) is valid only if the following relation holds:

$$(\beta_{5-})(\beta_{5+}) \neq 0 \tag{40}$$

We will assume throughout our calculations that this relation is valid.

In seismic applications it is usually reasonable to assume that the excitation, $u(t)$, is a scalar function, and we will do so. Define the scalar quantity:

$$\overline{\gamma} = \gamma\phi^{n^T}B \tag{41}$$

where γ is defined in eq.(32). Then the displacement of the equipment with respect to the ground, as a function of the ground motion becomes:

$$y(t) = \overline{\gamma}\int_0^t \nu(t,\theta)u(\theta)\,d\theta \tag{42}$$

The maximum inertial force acting on the secondary equipment is approximated as:

$$\widehat{F} = m_e\omega_e^2\widehat{y} \tag{43}$$

where $\widehat{y}$ is the maximum displacement of the equipment, driven by an uncertain input which is constrained by a convex model. We now evaluate $\widehat{y}$ for the Fourier-envelope convex model.

6.2 Maximum Force with the Fourier-Envelope Convex Model

Let us assume that $u(t)=0$ for $t<0$, and that $\int_0^\infty u^2(t)\,dt$ is bounded. Then the symmetrical Fourier transform pair is defined as:

$$u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{u}(\omega)\mathrm{e}^{-j\omega t}\,d\omega, \quad \overline{u}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty u(t)\mathrm{e}^{j\omega t}\,dt \tag{44}$$

The Fourier-envelope convex model is the set of input functions $u(t)$ for which the norm of the Fourier transform $\overline{u}(\omega)$ is contained in an envelope:

$$\mathcal{F}_{\mathrm{FE}} = \left\{u(t):\ |\overline{u}(\omega)|^2 \leq R^2(\omega)\right\} \tag{45}$$

The real and complex parts of $\overline{u}(\omega)$ are just the Fourier cosine and sine transforms of $u(t)$, which we denote $\overline{u}_c(\omega)$ and $\overline{u}_s(\omega)$ respectively:

$$\overline{u}(\omega) = \overline{u}_c(\omega) + j\overline{u}_s(\omega) \tag{46}$$

$\overline{u}_c(\omega)$ and $\overline{u}_s(\omega)$ are real functions:

$$\overline{u}_c(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t) \cos\omega\theta \, dt, \quad \overline{u}_s(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t) \sin\omega\theta \, dt \tag{47}$$

Substituting (46) into the first of eqs.(44) and recognizing that $u(t)$ is a real function one finds:

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\overline{u}_c(\omega) \cos\omega t - \overline{u}_s(\omega) \sin\omega t\right] \, d\omega \tag{48}$$

Substituting this into eq.(42) and changing the order of integration one finds the displacement of the equipment to be:

$$y(t) = \overline{\gamma} m_e \omega_e \int_{-\infty}^{\infty} \left[\overline{u}_c(\omega)\widetilde{\nu}_c(t,\omega) - \overline{u}_s(\omega)\widetilde{\nu}_s(t,\omega)\right] \, d\omega \tag{49}$$

where we define:

$$\widetilde{\nu}_c(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_0^t \widetilde{\nu}(t,\theta) \cos\omega\theta \, d\theta, \quad \widetilde{\nu}_s(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_0^t \widetilde{\nu}(t,\theta) \sin\omega\theta \, d\theta \tag{50}$$

To maximize $y(t)$ on $\mathcal{F}_{\rm FE}$ we note that the constraint on $\overline{u}(\omega)$, at each value of ω, is:

$$\overline{u}_c^2(\omega) + \overline{u}_s^2(\omega) \leq R^2(\omega) \tag{51}$$

The Cauchy inequality is now used to find the maximum displacement as:

$$\widehat{y}(t) = \max_{u\in\mathcal{F}_{\rm FE}} y(t) = \overline{\gamma} m_e \omega_e \int_{-\infty}^{\infty} R(\omega) \sqrt{\widetilde{\nu}_c^2(t,\omega) + \widetilde{\nu}_s^2(t,\omega)} \, d\omega \tag{52}$$

Writing this more explicitly, recalling that condition (40) is assumed to hold, and using eq.(43), the maximum inertial force acting on the secondary equipment is:

$$\widehat{F}(t) = \frac{m_e \omega_e^2 \phi_i^n \left(\phi^{nT} B\right)}{m_n \omega_n \sqrt{1-\zeta_n^2}\sqrt{1-\zeta_e^2}} \int_{-\infty}^{\infty} R(\omega) \sqrt{\widetilde{\nu}_c^2(t,\omega) + \widetilde{\nu}_s^2(t,\omega)} \, d\omega \tag{53}$$

This relation expresses the maximum inertial force exerted on the secondary equipment, which is allowed by the Fourier-envelope convex model of the seismic-uncertainty. This maximum is proportional to the equipment mass m_e and to the square of the natural frequency, ω_e, of the equipment-structure coupling, and inversely proportional to the modal mass and frequency of the dominant structural mode. The quantities $\widetilde{\nu}_c(t,\omega)$ and $\widetilde{\nu}_s(t,\omega)$ are functions of dimensionless coefficients depending on the the damping ratios of the equipment and of the structural mode, and on the ratio ω_n/ω_e of the structural to the equipment natural frequencies. The function $R(\omega)$ is the convex-model bound on the spectrum of the input, and is based on measured spectral variability of earthquakes. One can use this relation to assign values to the design parameters so as to assure that the inertial forces on the secondary equipment are always within acceptable limits.

7 Reliability of Axially-Loaded Shells With Initial Geometrical Imperfections

The examples discussed in sections 4 and 6 deal with uncertain external forces acting upon a system. In this section we will use a convex model to perform a reliability-analysis with respect to structural uncertainty. We consider an axially-compressed thin-walled shell with initial geometrical imperfections.

The shell length is L. The axial coordinate, along the length of the shell, is z, which we normalize as $\xi = \pi z/L \in [0, \pi]$. The azimuthal coordinate is $\theta \in [0, 2\pi]$. The deviation of an actual shell from the nominal shell dimension at point (ξ, θ) is $\eta(\xi, \theta)$. We represent the set of allowed imperfection-functions by the uniform-bound convex model:

$$\mathcal{F}_{\mathrm{UB}}(\widehat{\eta}) = \{\eta(\xi, \theta) : \ |\eta(\xi, \theta)| \leq \widehat{\eta}\} \tag{54}$$

The deviations from the nominal initial shell shape are uniformly bounded by $\widehat{\eta}$. Every imperfection-function, $\eta(\xi, \theta)$, whose magnitude nowhere exceeds $\widehat{\eta}$, is included in $\mathcal{F}_{\mathrm{UB}}$. One can view $\widehat{\eta}$ as a radial tolerance of the shells whose imperfections are represented by $\mathcal{F}_{\mathrm{UB}}$.

A typical question which arises in design-for-reliability is: how large a radial tolerance is acceptable, when the shell will bear static axial loads up to the value $\lambda_{\max}$?

Implicit in this question is a statement about the uncertainty in the actual shell shapes. If in fact the designer knows nothing about the geometrical imperfections other than the value of the radial tolerance to which the shells have been produced, then $\mathcal{F}_{\mathrm{UB}}$ is probably the most detailed representation of the range of possible shell shapes which can be justified by the available data. If additional information is available, such as spectral data about the spatial frequencies of the imperfections, then other convex models would be appropriate. Various more detailed convex models for this purpose are discussed in [2, 4, 5].

We will proceed with the simple uniform-bound convex model. The design question can be formulated as follows. The design-parameter is $\widehat{\eta}$, the radial tolerance. Denote by $\mu(\widehat{\eta})$ the least buckling load of any shell in $\mathcal{F}_{\mathrm{UB}}(\widehat{\eta})$. Then determine the greatest value of the radial tolerance, $\widehat{\eta}$, for which the least buckling load, $\mu(\widehat{\eta})$, exceeds the maximum load, $\lambda_{\max}$.

The mechanical analysis of geometrically imperfect shells is most conveniently done when the imperfections are expressed in terms of their Fourier coefficients. Let $x(\eta)$ be a vector of the dominant Fourier coefficients of $\eta(\xi, \theta)$. Let x^0 be the vector of Fourier coefficients of the nominal shell shape. Let $\Psi(x^0)$ be the buckling load of this nominal shell, and $\Psi(x^0 + x)$ be the buckling load of a shell with initial imperfections whose Fourier coefficients are x. For small imperfections we can expand $\Psi(x^0 + x)$ as:

$$\Psi\left[x^0 + x(\eta)\right] = \Psi(x^0) + x^T(\eta) \left.\frac{\partial \Psi}{\partial x}\right|_{x=x^0} \tag{55}$$

Some manipulations lead to the following expression for the reduced buckling load due to the imperfection function $\eta(\xi, \theta)$:

$$\Psi\left[x^0 + x(\eta)\right] = \Psi(x^0) + \int_0^{2\pi} \int_0^{\pi} \eta(\xi, \theta) S(\xi, \theta)\, d\xi\, d\theta \tag{56}$$

where $S(\xi, \theta)$ is a combination of trigonometric functions with coefficients which depend on the elements of the vector $\partial \Psi(x = x^0)/\partial x$. See [4, 5].

Examination of eq.(56) shows that the greatest reduction in the buckling load is obtained from the imperfection-function which switches between its extreme values, $+\widehat{\eta}$ and $-\widehat{\eta}$, as $S(\xi, \theta)$ changes sign from negative to positive. The minimum buckling, load for shells whose radial tolerance is $\widehat{\eta}$, is:

$$\mu(\widehat{\eta}) = \min_{\eta \in \mathcal{F}_{\mathrm{UB}}} \Psi\left[x^0 + x(\eta)\right] \tag{57}$$

$$= \Psi(x^0) - \widehat{\eta} \int_0^{2\pi} \int_0^{\pi} |S(\xi, \theta)| \, d\xi \, d\theta \tag{58}$$

This relation expresses the buckling load of the weakest shell from among the ensemble of shells whose radial tolerance is $\widehat{\eta}$. Additionally, it is based on mechanical data expressing the imperfection-sensitivity of the buckling load, which appears in the function $S(\xi, \theta)$. Eq.(58) is derived for small imperfections, and is therefore linear in the parameter $\widehat{\eta}$.

Eq.(58) underlies the convex-modelling assessment of the reliability of the uncertain shell. The shell uncertainty is expressed by $\widehat{\eta}$ and the range of performance — embodied in the least buckling load — is given by $\mu(\widehat{\eta})$. One chooses the radial tolerance to assure that the maximum axial load does not exceed the least buckling load:

$$\lambda_{\max} < \mu(\widehat{\eta}) \tag{59}$$

Uncertainty plays a central role in this analysis: $\mathcal{F}_{\mathrm{UB}}$ represents a set of shells, any one of which could occur. Any given physical shell with tolerance not exceeding $\widehat{\eta}$ is represented by one of the imperfection functions in $\mathcal{F}_{\mathrm{UB}}$; which one, one does not know. The shell is 'reliable' in the sense of our non-probabilistic model of uncertainty when (59) is satisfied.

One must understand clearly that, while uncertainty in the shell shapes is fundamental to this analysis, there is no frequency or likelihood information, either in the formulation of the convex model or in the concept of reliability. It might be useful, for example, to assess the reliability of a given radial tolerance by a quantity such as:[6]

$$r = 1 - \frac{\lambda_{\max}}{\mu(\widehat{\eta})} \tag{60}$$

When r is close to unity, the maximum load is far less than the least load-bearing capacity; the system is 'reliable' in the non-probabilistic sense. As r approaches zero, the maximum load approaches the least buckling load, and failure becomes more imminent. However, unlike in a probabilistic analysis, r has no connotation of likelihood. We have no rigorous basis for evaluating how likely failure may be; we simply lack the information, and to make a judgement would be deceptive and could be dangerous. There may definitely be a likelihood of failure associated with any given radial tolerance. However, the available information does not allow one to assess this likelihood with any reasonable accuracy.

8 Summary

We have stressed the following ideas in this paper:

1. The modelling of uncertainty can be either probabilistic or non-probabilistic. We have employed several convex models to implement the latter option.

2. Many authors, from both philosophical and technological areas, have noted that the details of probability distributions are often difficult to verify or justify with concrete data.

3. Analysis and design for high probabilistic reliability are very sensitive to small inaccuracies in the tails of the probability density function.

[6] I am indebted to Prof. Cempel for this suggestion.

4. Convex models are structurally different from probabilistic models. In particular, they have no probability densities. Instead, they use information about the uncertainty to specify the structure of sets of uncertain events.

5. A theorem is presented which compares the output of a system driven by a stochastic process, with the output of the same system driven by a convex model which is homologous to the stochastic process. The theorem suggests that input uncertainties which appear quite similar when modelled by either a stochastic or a convex model can lead to output uncertainties which are significantly different.

6. We have examined three examples, covering both input uncertainties and structural-geometric uncertainties.

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