73. **Supply network,** (p.256).

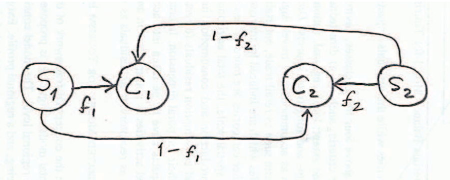

Figure 16: Topology for two sources and two consumers, problem 73.

(a) Consider a network that supplies a good (e.g. water, electricity or cookies) to consumers. The network has two suppliers and two consumers. Source i produces quantity s_i and supplies a fraction f_i to consumer i, and a fraction $1-f_i$ to consumer j, as in fig. 16. Source j acts similarly. Each consumer consumes whatever is supplied. Thus the consumption by consumer i is:

$$c_i = f_i s_i + (1 - f_j)s_j \tag{299}$$

where $i = 1, 2$ and $j = 3 - i$.

There is fractional-error uncertainty in the source properties:

$$\mathcal{U}(h) = \left\{ f_i, s_i : \left| \frac{f_i - \widetilde{f}_i}{\widetilde{f}_i} \right| \leq h, \ f_i \in [0, 1], \ \left| \frac{s_i - \widetilde{s}_i}{\widetilde{s}_i} \right| \leq h, \ s_i \geq 0, \ i = 1, 2 \right\}, \quad h \geq 0 \tag{300}$$

We require that each consumer be within δ of a specified value, $\overline{c}$:

$$|c_i - \overline{c}| \leq \delta \tag{301}$$

Derive an explicit expression for the inverse of the robustness function for consumer i. Evaluate and compare the robustnesses for $\overline{c} = 1$, $\widetilde{f} = 1/2$ and $\widetilde{s} = 0.9$ or 1.0. Which option is preferred? Why, and what does this mean?

(b) Modify part 73a as follows. The nominal consumption by each consumer is $\widetilde{c}$ but the actual consumption is:

$$c = \widetilde{c} + \varepsilon \tag{302}$$

where ε is an exponentially distributed random variable whose pdf is $p(\varepsilon) = \lambda e^{-\lambda\varepsilon}$, $\varepsilon \geq 0$. Derive an explicit algebraic expression for the probability that $c \leq \overline{c}$ where $\overline{c}$ is a known positive value greater than $\widetilde{c}$.

(c) Continuing part 73b, consider uncertainty in the exponential coefficient, λ, represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{w} \right| \leq h \right\}, \quad h \geq 0 \tag{303}$$

where $\widetilde{\lambda}$ and w are known positive values. We require that the probability that $c \leq \overline{c}$ be less than δ where $0 < \delta < 1$. Derive an explicit algebraic expression for the robustness.

(d) Continuing part 73c, consider two different designs with values $\widetilde{\lambda}_i$ and w_i, for $i = 1$ and 2. Based on the robustness function, derive an explicit algebraic expression for the values of δ for which you prefer system 1.

(e) A particular consumer (you, perhaps) is supplied by N sources resulting in consumption equal to:

$$c = \sum_{i=1}^{N} f_i s_i \tag{304}$$

The fractions f_i and source terms s_i are uncertain:

$$\mathcal{U}(h) = \left\{ f, s: \ f_i \ge 0, \ \left| \frac{f_i - \widetilde{f}_i}{\widetilde{f}_i} \right| \le h, \ s_i \ge 0, \ \left| \frac{s_i - \widetilde{s}_i}{\widetilde{s}_i} \right| \le h, i = 1, \ldots, N \right\}, \quad h \ge 0 \quad (305)$$

We require that the consumption be no less than the critical value $\overline{c}$:

$$c \ge \overline{c} \quad (306)$$

Derive an explicit algebraic expression for the robustness function.

(f) Repeat part 73e with the modification that the fractions, f_i, are positive and known for sure and the source terms are uncertain according to:

$$\mathcal{U}(h) = \left\{ s: \ (s - \widetilde{s})^T W^{-1} (s - \widetilde{s}) \le h^2 \right\}, \quad h \ge 0 \quad (307)$$

where $\widetilde{s}$ is a known vector and W is a known, positive definite, real, symmetric matrix. The performance requirement is eq.(306). Derive an explicit algebraic expression for the robustness function.

(g) Consider the following modification of the 2-source and 2-consumer network, in which we introduce a mutual commitment. Under ordinary conditions, each source supplies a single consumer at each discrete time step:

$$c_1(t) = s_1, \quad c_2(t) = s_2, \qquad t = 0, 1, 2, \ldots \quad (308)$$

Each consumer has its own private supply, and each consumer requires a positive consumption $\overline{c}_i$, $i = 1, 2$.

However, the consumers have mutual commitments. If consumer i loses its supply at some time step, then in the next time step consumer j is committed to supply consumer i with a fraction γ of i's requirement, $\overline{c}_i$, though j cannot supply more than s_j provides.

Suppose that at some time step, call it $t = 0$, consumer 1 loses its supply, so the consumption in this step is:

$$c_1(0) = 0, \quad c_2(0) = s_2 \quad (309)$$

In the next time step, consumer 2 must transfer to consumer 1 a part of 2's supply so the consumption is:

$$c_1(1) = \min(s_2, \gamma \overline{c}_1), \quad c_2(1) = (s_2 - \gamma \overline{c}_1)^+ \quad (310)$$

where $x^+ = x$ if $x \ge 0$ and equals zero otherwise.

The sources are uncertain according to a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ s: \ s_i \ge 0, \ \left| \frac{s_i - \widetilde{s}_i}{w_i} \right| \le h, i = 1, 2 \right\}, \quad h \ge 0 \quad (311)$$

where $\widetilde{s}_i$ and w_i are known positive constants.

Derive an explicit algebraic expression for the robustness of consumer 2 at step 1.

(h) Using the robustness function from part 73g, consider the following two commitment situations, (γ, w_2) and (γ', w_2'), where:

$$\gamma' < \gamma, \quad w_2' > w_2 \quad (312)$$

The 'prime' configuration entails lower commitment by consumer 2, but greater uncertainty in consumer 2's source. Use the robustness function to discuss the values of consumer 2's required consumption, $\overline{c}_2$, for which 2 prefers the 'prime' configuration.