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# Measured Averages and Inferred Extremes: Info-Gap Analysis of Deep Uncertainty

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## Contents

**1 Introduction** **2**

**2 Info-Gap Robustness and Opportuneness in Response to Deep Uncertainty** **3**

**3 Example: Imposing Speed Limits** **5**
3.1 Info-Gap Model of Uncertainty . . . . 6
3.2 Robustness Against Uncertainty . . . . 7

**4 Example: Inferring the Level of Economic Activity** **11**
4.1 Info-Gap Model of Uncertainty . . . . 11
4.2 Robustness Against Uncertainty . . . . 12
4.3 Opportuneness From Uncertainty . . . . 13

**5 Example: Statistical Inference** **15**
5.1 Info-Gap Model of Uncertainty . . . . 15
5.2 Robustness Against Uncertainty . . . . 16
5.3 Opportuneness From Uncertainty . . . . 18

**6 Robustness as a Proxy for the Probability of Success** **20**
6.1 Speed Limits . . . . 21
6.2 Economic Activity . . . . 22
6.3 Statistical Inference . . . . 22

**7 Conclusion** **23**

**8 References** **25**

**A Maximum Acceleration** **26**

**Abstract** Averages are measured for diagnosis, prediction, or surveillance. However, averages reveal nothing about fluctuations, and extreme values may be more significant than the average. The analyst can choose decision variables: path length and other parameters. This paper explores the choice of decision variables to achieve robustness against pernicious uncertainty when interpreting an average, in face of uncertain fluctuations of

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the averaged variable. We also explore the choice of decision variables to achieve opportuneness from propitious uncertainty. Trade offs and "trade-ons" between robust and opportune decision variables are identified. Three examples are developed: enforcing speed limits; inferring levels of economic activity; statistical hypothesis testing. We use concepts of robustness and opportuneness from info-gap decision theory. We also explore the relation between the probability of success and the non-probabilistic robustness to uncertainty, demonstrating conditions where robustness is a proxy for probability.

**Keywords** averages, extremes, info-gaps, uncertainty, robustness, opportuneness, speed limits, economic activity, statistical inference.

# 1 Introduction

If you travel on a highway at 75 km/h for 59 minutes and 150 km/h for 1 minute, then your average speed is 76.25 km/h; a moderate average, but not safe driving. A drinking binge just one day a month would not look like excessive alcohol intake when averaged over time [8], but it's hardly healthy behavior. Some faces are more attractive than others, but it is found that the average values of facial parameters are widely considered the best [21]. The average mechanical stress along a beam is not a good measure of safety because local beam imperfections or local loading may cause dangerous local stress concentrations [6]. A country's annual gross domestic product is an average over regions which may differ greatly. For instance, the volume index of GDP/capita in 2006[1] for Australia, Japan, Republic of Korea and Russia are: 125.1, 112.7, 89.3, and 41.3. However, the ranges between small regions of each country, from minimum to maximum GDP/capita as percent of national GDP/capita in 2006, for AU, JN, RoK and RU are: (80.1, 126.9), (65.5, 181.5), (63.3, 217.5), (15.6, 676.5) [20]. Two students with the same 4-year average grade may display very different trends or fluctuations in time and among disciplines.

Deviations from the mean are not necessarily pernicious. Extreme deviations from the average may be propitious opportunities for better-than-anticipated outcomes. Quick stock market profits depend on detecting drops and peaks in prices. Impetuous drivers can be caught only by detecting their transient episodes of speeding. Curing binge drinkers may be most effective on the day after their binge.

In this paper we will discuss the use of averages for regulating highway speed, for inferring levels of economic activity, and for making statistical inference. In all cases, however, it is the extreme values that concern the policy maker or planner, and these extremes are deeply uncertain.

The theory of probability is widely used and can be highly useful. Its usefulness derives partly from its underlying assumptions, especially the assumption that relevant probability distributions are reliably known. The present paper explores uncertainty in the shapes

[1] OECD=100 in year 2000 at 2000 price levels and PPPs.

of functions. The probabilistic representation of functional uncertainty is difficult for two reasons. First, it is mathematically difficult to represent probabilities in a function space, and this requires various assumptions about the processes involved. Second, we study processes about which knowledge is sparse, so extreme values are elusive and functional uncertainty abounds. Extreme value theory does not require knowledge of probability distributions. However, extreme value theory assumes that the random variables are independent and identically distributed, among other assumptions [10, p.261]. We will consider situations where this assumption is unjustified. We will employ info-gap decision theory, as applied elsewhere to related issues [6]. Info-gap theory is a non-probabilistic method for modeling and managing uncertainty [2].

Average values disclose nothing about deviations from the average. However, by treating those deviations as uncertain, and by evaluating the robustness to that uncertainty using limited additional information, the policy maker or strategic planner can assess and augment the usefulness of an average for indicating or reducing extreme deviations. That is, acknowledging the deep uncertainty of deviations, and managing that uncertainty by evaluating the robustness to uncertainty, supports interpretations of the average in terms of extreme deviations.

The present paper employs the info-gap formulations of robustness and opportuneness that have been applied in many situations for protection against pernicious surprise or for exploitation of propitious surprise. One innovation of this paper is that the robustness and opportuneness functions are used for inference about extreme deviations from the observed average. We are not protecting against or exploiting surprise as in many prior applications of info-gap theory. Instead, we are using the info-gap robustness and opportuneness functions to disclose various properties of extreme deviations from the average. These functions are formulated in section 2.

We will consider three examples in depth. The first example, in section 3, entails setting highway speed limits, where the implicit goal is to constrain speeding that is far in excess of the legal limit. The second example, in section 4, involves inference about low levels of economic activity based only on observed average levels. The third example, in section 5, involves statistical hypothesis testing. Section 6 explores the use of the non-probabilistic robustness function as a proxy for the probability of satisfying specified conditions. When this proxy property holds, one can enhance or maximize the probability of success by enhancing or maximizing the robustness without any probabilistic information about the processes involved. Section 7 is a brief conclusion. Some generic properties that are common to all three examples are developed and discussed elsewhere [6].

# 2 Info-Gap Robustness and Opportuneness in Response to Deep Uncertainty

Many applications, including those in sections 3–5, depend upon functions that vary over space or time. Information about these functions is available, but there remains deep uncertainty about their shapes. This section formulations the info-gap functions of robustness and opportuneness whose use is demonstrated in later sections.

Humanity has always faced deficient or erroneous information. However, the systematic study of uncertainty began only in the early 17th century, leading to concepts of probability [13]; statistical inference emerged only in the 19th century [22, 24]. Diverse models of uncertainty emerged in the 20th century. Lukaczewicz developed 3-valued logic in 1917 [16], Wald formulated a modern version of min-max in 1945 [25], and in 1965 Zadeh introduced his work on fuzzy logic [27]. Many other theories, including P-boxes [12], lower previsions [18, 26], Dempster-Shafer theory [10, 23], generalized information theory [14] and info-gap decision theory [2, 4] have continued to sprout up.

These theories for modeling and managing uncertainty differ substantially from one another, and we won't elaborate their differences. However, these theories are all axiomatically distinct from the theory of probability whose axiomatization was established by Kolmogorov [15]. For example, probability density functions must be normalized to unity, but this does not hold for fuzzy belief functions. Similarly, the probabilistic representation of uncertainty employs real-valued scalar functions, while P-boxes and info-gap models of uncertainty are set-valued functions.

The choice of an uncertainty model should match the type of knowledge and ignorance that must be represented and managed. In the present paper we confront extremely sparse information, and for this purpose minimalistic info-gap models of uncertainty are suitable.

Our basic notation is as follows.

$x$ is an *independent scalar variable* denoting time or spatial location. In the spatial context we will assume that space is 1-dimensional, for instance position on a road or on a path through 2- or 3-dimensional space. $x$ varies on the interval $[0, D]$.

$v(x)$ is a *substantive scalar function* denoting, for example, speed or mechanical load at location $x$, or quantity of consumption per unit time at time $x$, or GDP per unit distance at location $x$, etc.

The *performance requirement* is expressed by a constraint on the substantive function $v(x)$. That is, the planner or analyst requires that:

$$F[v(x)] \leq v_{\text{c}} \tag{1}$$

where $F[\cdot]$ is a known scalar-valued function. The value $v_{\text{c}}$ is a critical requirement, fixed by a regulator, or by a client, or by other means. It is acceptable for the function $F[v(x)]$ to be less than the critical value, but exceeding $v_{\text{c}}$ is prohibited. The robustness function, to be defined shortly, establishes a degree of confidence in satisfying the critical requirement in eq.(1). The function $F$ is called the *performance function.* It may entail integration over the domain $[0, D]$, or some other operation such as maximization over the domain as in section 3.2. Alternatively, $F(\cdot)$ may operate only on part of the domain, or it may be evaluated at one or more specific values of $x$ such as the midpoint or end of the domain.

$v_{\text{c}}$ is the largest acceptable value of $F[v(x)]$, but smaller values are acceptable and even desirable. In some situations the planner or analyst aspires to achieve a wonderfully small value, $v_{\text{w}}$, which is less than $v_{\text{c}}$. In this case the performance aspiration is to attempt to satisfy eq.(1) with $v_{\mathrm{w}}$ instead of $v_{\mathrm{c}}$, but not to require this condition. The opportuneness function, to be defined soon, assesses the feasibility of enabling, though not necessarily guaranteeing, the windfall value $v_{\mathrm{w}}$.

Finally, the substantive function, $v(x)$, is uncertain, as expressed by an *info-gap model of uncertainty* [2]. An info-gap model of uncertainty is an unbounded family of nested sets, $\mathcal{U}(h)$, of the uncertain entity, $v(x)$. An info-gap model expresses the uncertain knowledge about the substantive function, but no probability distributions are involved; the info-gap model is non-probabilistic as discussed in detail elsewhere [1].

All info-gap models obey the nesting axiom that asserts that the sets become more inclusive as the value of $h$ increases:

$$h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{2}$$

This axiom endows $h$ with its meaning as an horizon of uncertainty, and its value is unknown and unbounded though non-negative.

In many applications the info-gap model also obeys the contraction axiom that asserts that there is a known substantive function $\widetilde{v}(x)$ as the only possibility in the absence of uncertainty:

$$\mathcal{U}(0) = \{\widetilde{v}(x)\} \tag{3}$$

The info-gap models of sections 3 to 5, eqs.(6), (20) and (39), all obey the nesting axiom, but only the latter two info-gap models also obey contraction. The info-gap model of eq.(6) does not obey contraction because of the one-sided asymmetrical constraint on the substantive functions.

For simplicity we will assume that the sets of the info-gap model are closed sets.

The *robustness to uncertainty* is the greatest horizon of uncertainty, $h$, up to which the performance requirement in eq.(1) is guaranteed to be satisfied for all realizations of the uncertain substantive function in the uncertainty set $\mathcal{U}(h)$. The formal definition of the robustness is:

$$\widehat{h}(v_{\mathrm{c}}) = \max \left\{ h : \left( \max_{v \in \mathcal{U}(h)} F[v(x)] \right) \leq v_{\mathrm{c}} \right\} \tag{4}$$

The *opportuneness from uncertainty* is the least horizon of uncertainty, $h$, at which the performance aspiration — not requirement — in eq.(1), with $v_{\mathrm{w}}$ rather than $v_{\mathrm{c}}$, is satisfied for at least one realization of the uncertain substantive function in the uncertainty set $\mathcal{U}(h)$. The formal definition of the opportuneness is:

$$\widehat{\beta}(v_{\mathrm{c}}) = \min \left\{ h : \left( \min_{v \in \mathcal{U}(h)} F[v(x)] \right) \leq v_{\mathrm{w}} \right\} \tag{5}$$

# 3 Example: Imposing Speed Limits

The regulation of car traffic on roadways draws the attention of a vast proportion of humanity on a daily basis, and has also been the focus of scholarly research. For instance, Delle Monache *et al.* [9] study the optimal control of road traffic by employing variable speed limits, noting (and managing) the substantial challenges in solving the flow patterns. They compare alternative strategies for traffic control, assuming that drivers obey speed limits. In this example we will relax that assumption in a much simpler situation that avoids the flow-dynamic complexities. We focus on the choice of the speed limit and of the size of the road section in which average speeds are monitored.

The speed of a car as a function of position along the road is denoted $v(x)$. The car passes a sensor at position $x = 0$ and the distance to the next sensor is $D$. Let $v_0$ denote the speed when passing the first sensor. This may in fact be an actual measurement on a specific car that is then tracked. The next sensor will detect the passing of the car and determine the time elapsed, $T$, since the car passed the previous sensor. The average speed of the car between the sensors is $D/T$. A traffic fine will be imposed if the average speed exceeds the legal speed limit.

Considerations of road safety indicate that cars should not exceed a safe speed $v_s$ at any time between the sensors. Recognizing that drivers will exceed speed limits, the legal speed limit is set at a lower value, $v_\ell$. That is, the traffic law requires $v(x) \le v_\ell$ for all values of $x$. The disparity between $v_s$ and $v_\ell$ can be viewed as part of a strategic interaction between drivers and regulators.

The problem confronting the regulator is that the average speed between sensors does not reflect local variations of speed. The questions we ask relate to robustness against this uncertainty. Is the determination of speed compliance robust to uncertainty in the speed profile between sensors? How does this robustness vary with the distance, $D$, between sensors and with the legal speed limit $v_\ell$? What is the robustness at the safe speed, $v_s$, and how does this change as the disparity between $v_\ell$ and $v_s$ increases? What are the implications for choice of $D$ and $v_\ell$, for given $v_s$, to enhance robustness to uncertain variations in speed?

## 3.1 Info-Gap Model of Uncertainty

We will assume positive car velocities at all positions along the road. Thus position, $x$, increases monotonically with time, $t$, so we can express velocity as a function of time, $v(t)$. This is important because we will need to consider acceleration. The car passes the first sensor at time $t = 0$ and reaches the second sensor at time $T$.

The regulator imposes the speed limit $v_\ell$ but does not know the extent to which drivers will exceed this value. Furthermore, the regulator estimates the typical potential for acceleration of cars at the positive value $\widetilde{a}$ [17]. Cars typically display zero acceleration, but their actual acceleration, $\dot{v}(t)$, relative to the estimate is uncertain. The following info-gap model quantifies these uncertainties non-probabilistically and also posits positive velocities:

$$\mathcal{U}(h) = \left\{ v(t) : \ v(t) > 0, \quad \frac{v(t) - v_\ell}{v_\ell} \le h, \quad \left| \frac{\dot{v}(t)}{\widetilde{a}} \right| \le h \right\}, \quad h \ge 0 \tag{6}$$

$\mathcal{U}(h)$ is the set of all positive speed profiles, $v(t)$, whose fractional deviation above the speed limit, $v_\ell$, is no greater than the horizon of uncertainty, $h$, and whose absolute acceleration, relative to the estimate $\widetilde{a}$, is no greater than $h$. The value of $h$ is unknown and unbounded, so the info-gap model is not a single set, but rather an unbounded family of nested sets of possible speed profiles. This means that there is no known worst case. Furthermore, the info-gap model is a non-probabilistic quantification of uncertainty.

The conceptual tool that one employs in assessing extreme responses should match the available knowledge and its lacunae, as well as the goals that one seeks to attain. The info-gap model of eq.(6) specifies what we know, and what we don't know, about the speed profile function $v(t)$. It is important to stress that our very limited knowledge is entirely non-probabilistic. For instance, we have no basis for supposing that velocities at distinct times are statistically independent and identically distributed. In fact, it is quite plausible to suppose that neither of these properties holds, either for any specific driver during a journey, or between different drivers. We are concerned with extreme values that the speed profile may obtain, or exceed, but the fundamental probabilistic assumptions that underlie extreme value distributions — independent and identically distributed random variables [19, p.261] — are not warranted given the available knowledge. Extreme value distributions would be useful, but they are not accessible given the state of our knowledge. This motivates the info-gap analysis of robustness to uncertainty.

## 3.2 Robustness Against Uncertainty

The robustness to uncertainty in the speed profile is the greatest horizon of uncertainty, $h$, up to which all speed profiles in the uncertainty set $\mathcal{U}(h)$ never exceed a safe speed, $v_{\mathrm{s}}$, throughout the travel between the sensors. Let $T$ denote the time of travel between the sensors, whose value has yet to be determined. The definition of the robustness as a function of the safe speed, $v_{\mathrm{s}}$, is [2, 4]:

$$\widehat{h}(v_{\mathrm{s}}) = \max \left\{ h : \left( \max_{v \in \mathcal{U}(h)} \max_{t \in [0,T]} v(t) \right) \leq v_{\mathrm{s}} \right\} \tag{7}$$

Let $m(h, t)$ denote the maximum speed at horizon of uncertainty $h$ and at time $t$. That is, $m(h, t)$ is the inner double maximum in the definition of the robustness function, eq.(7).

The distance between sensors, $D$, is fixed. The car passes the 1st sensor at time $t = 0$ and the time of arrival at the 2nd sensor, $T$, depends on the car's speed profile. If the car accelerates greatly its speed rises rapidly and its arrival at the 2nd sensor occurs quickly, allowing little time for acceleration. This may suggest that its final speed is not maximal, from among all possible speed profiles, if the acceleration is maximal. Nonetheless we can readily understand that the maximum speed occurs at the 2nd sensor and is obtained when the car accelerates maximally, even though this minimizes the transit time. The proof appears in appendix A.

If the car's velocity exceeds the speed limit at the first sensor then a fine would be imposed at that time and the car would no longer interest the regulator. Thus let us suppose that the velocity when passing the 1st sensor, $v_0$, is known to the regulator and does not exceed the speed limit:

$$v_0 \leq v_\ell \tag{8}$$

This alters the info-gap model of eq.(6) by adding the further constraint on its elements that $v(0) = v_0$.

Assuming that the constants $v_\ell$, $v_0$ and $\widetilde{a}$ are positive, the maximum speed at time $t$ in the definition of the robustness is the lesser of the two maxima obtained from the velocity and acceleration constraints of the info-gap model:

$$m(h,t) = \min\{(1+h)v_\ell,\ v_0 + \widetilde{a}ht\} \tag{9}$$

From eq.(8), $m(h,t)$ equals the second term in eq.(9) at short times, and switches to the first term at the time, denoted $t_\mathrm{s}(h)$, at which the two velocities are equal:

$$(1+h)v_\ell = v_0 + \widetilde{a}ht \quad \Longrightarrow \quad t_\mathrm{s}(h) = \frac{(1+h)v_\ell - v_0}{\widetilde{a}h} \tag{10}$$

Thus:

$$m(h,t) = \begin{cases} v_0 + \widetilde{a}ht & \text{if} \quad t \le t_\mathrm{s}(h) \\ (1+h)v_\ell & \text{else} \end{cases} \tag{11}$$

The distance between the sensors is known and equals $D$, so the time of arrival at the 2nd sensor, $T$, is the solution of:

$$D = \int_0^T m(h,t)\,\mathrm{d}t \tag{12}$$

Suppose that the 2nd sensor is reached no later than $t_\mathrm{s}(h)$:

$$T \le t_\mathrm{s}(h) \tag{13}$$

In this case, the arrival time is the solution for $T$ of:

$$D = \int_0^T (v_0 + \widetilde{a}ht)\,\mathrm{d}t = v_0 T + \frac{1}{2}\widetilde{a}hT^2 \tag{14}$$

Solving for $T$ yields two roots, only one of which is positive:

$$T_\mathrm{a}(h) = \frac{-v_0 + \sqrt{v_0^2 + 2\widetilde{a}hD}}{\widetilde{a}h} \tag{15}$$

where the subscript 'a' denotes that this is the arrival time if $v_0 + \widetilde{a}ht$ (velocity from maximum acceleration), is the minimum in eq.(9). Recall that this solution for $T$ depends on the supposition in eq.(13).

If the supposition in eq.(13) does not hold then the time of arrival at the 2nd sensor, $T$, is greater than $t_\mathrm{s}(h)$. However, in this case we don't need to know the value of $T$ because the final velocity does not depend on it, as stated in the 2nd line of eq.(11).

Thus, from eq.(11), the inner double-maximum in the definition of the robustness, eq.(7), is:

$$m(h,T) = \begin{cases} v_0 + \widetilde{a}hT_\mathrm{a}(h) & \text{if} \quad T_\mathrm{a}(h) \le t_\mathrm{s}(h) \\ (1+h)v_\ell & \text{else} \end{cases} \tag{16}$$

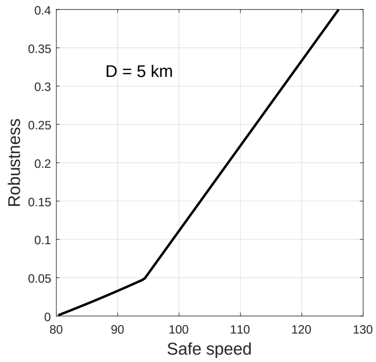


Figure 1: Robustness vs safe speed for $D = 5$ km. $v_0 = 80$ km/h, $v_\ell = 90$ km/h, $\widetilde{a} = 5184$ km/h$^2$.

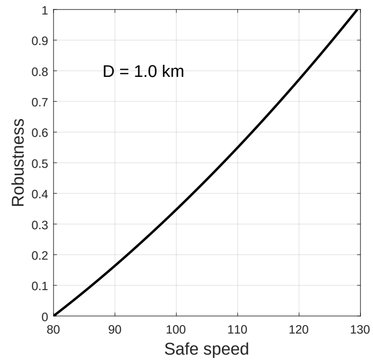


Figure 2: Robustness vs safe speed for $D = 1$ km. $v_0$, $v_\ell$ and $\widetilde{a}$ as in fig. 1.

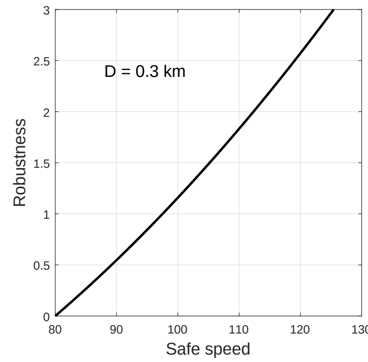


Figure 3: Robustness vs safe speed for $D = 0.3$ km. $v_0$, $v_\ell$ and $\widetilde{a}$ as in fig. 1.

We note that $m(h, T)$ is the functional inverse (not algebraic inverse) of the robustness function, $\widehat{h}(v_s)$. That is, a plot of $h$ vs $m(h, T)$ is the same as a plot of $\widehat{h}(v_s)$ vs. $v_s$.

Figs. 1–3 show robustness curves based on eq.(16) for 3 different values of the distance between the sensors. In fig. 1 we see the transition between the two rows of eq.(16). The lower part of the curve results from the upper line of eq.(16), and the upper part of the curve results from the lower line of eq.(16). The robustness curves in figs. 2 and 3 result entirely from the upper line of eq.(16) because the values of $T_a(h)$, eq.(15), are smaller for these shorter distances, while $t_s(h)$ in eq.(10) does not change with $D$.

Eqs.(8) and (9) show that the anticipated velocity is $v_0$ in the absence of uncertainty ($h = 0$). If this predicted velocity is chosen as the safe speed, $v_s$, then the robustness is zero, as seen in all three figures. This is the zeroing property observed in all info-gap robustness functions: predicted outcomes have no robustness against uncertainty in the data and models upon which the predictions are based.

The positive slopes of the curves in figs. 1–3 show that the robustness increases (which is good) as the safe speed is raised (which is undesirable). This expresses the irrevocable trade off between robustness and outcome requirement: More demanding outcomes (lower $v_s$) are less robust to uncertainty (lower $\widehat{h}$).

Fig. 4 shows robustness curves for three different speed limits, $v_\ell = 85, 90$ and $95$ km/h. The middle curve is the same as fig. 1. The curves coincide at low safe speeds, where the robustness follows the upper line in eq.(16) which does not depend on the speed limit, $v_\ell$. But then the curves diverge at greater safe speeds when the robustness shifts to the lower line in eq.(16). It is significant that the robustness improves as the speed limit is decreased.

We can summarize the conclusions from figs.1–4 as follows. First recall that $m(h, T)$, thought of as a function of $h$, is the functional inverse (not algebraic inverse) of the robustness function, $\widehat{h}(v_s)$. Thus a derivative of $m(h, T)$ with respect to $D$ or $v_\ell$ has the opposite sign as the same derivative of $\widehat{h}(v_s)$. Employing eq.(16), and as demonstrated in fig. 4, we

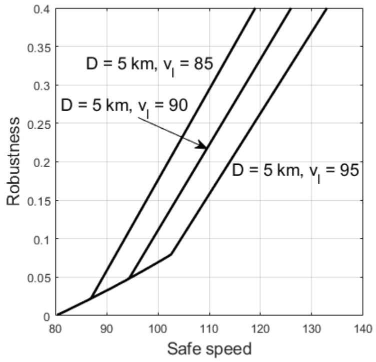


Figure 4: Robustness vs safe speed. $v_0 = 80$ km/h, $\widetilde{a} = 5184$ km/h$^2$.

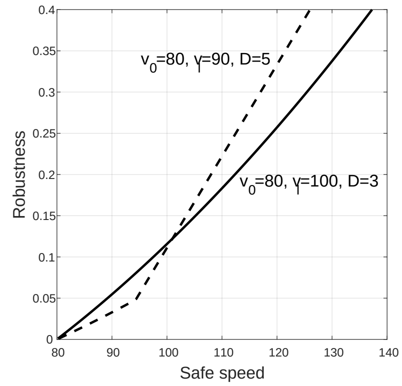


Figure 5: Robustness vs safe speed. $(v_\ell, D) = (90, 5)$ (dash). $(v_\ell, D) = (100, 3)$ (solid).

see that the robustness decreases as the speed limit is enlarged:

$$\frac{\partial m(h,T)}{\partial v_\ell} \geq 0 \quad \Longrightarrow \quad \frac{\partial \widehat{h}(v_{\rm s})}{\partial v_\ell} \leq 0 \tag{17}$$

Likewise, based on eq.(16) and as seen in figs.1–3, the robustness decreases as the domain get larger:

$$\frac{\partial m(h,T)}{\partial D} \geq 0 \quad \Longrightarrow \quad \frac{\partial \widehat{h}(v_{\rm s})}{\partial D} \leq 0 \tag{18}$$

Eqs.(17) and (18) show that changes of $v_\ell$ and $D$ in different directions have conflicting influences on the robustness, as we now discuss.

Fig. 5 shows robustness curves for two combinations of the speed limit, $v_\ell$, and the sensor separation distance $D$. These robustness curves cross one another, which presents a dilemma to the regulator in choosing between these design options. The design $(v_\ell, D) = (100, 3)$ (solid curve) is more robust than $(v_\ell, D) = (90, 5)$ (dashed curve) at low safe speeds (below about 102 km/h), while $(90, 5)$ is more robust than $(100, 3)$ at greater safe speeds. Neither design is robust-dominant over the other. The choice between them depends on the regulator's choice of the safe speed, and the judgment of how much robustness is necessary. This intersection between the robustness curves entails the potential for a reversal of preference between these two design options.

We can understand the intersection of the robustness curves in fig. 5 as expressing the conflicting tendencies demonstrated in figs. 1–3 and fig. 4. Figs. 1–3 show that the robustness increases as the domain gets smaller, while fig. 4 shows that the robustness decreases as the maximal element gets greater. The solid robustness curve in fig. 5 has a smaller domain and a larger maximal element than the dashed robustness curve. The tendencies asserted in the previous figures are in conflict, and the robustness curves intersect.

# 4 Example: Inferring the Level of Economic Activity

In the speed-limit example of section 3 the average is used to deter or modify behavior. We now consider inference from an observed average in support of decision making. We will see that the assessment of robustness in the inference problem is structurally similar to the behavior-deterrence example of section 3. We will also consider opportuneness from propitious uncertainty.

The average GDP per capita in a heterogenous region is used to determine the degree of public support to the region. Suppose the goal of the public support is to assure that the lowest percentile exceeds a specified minimal critical value. If the average is above this critical value, and the local GDP per capita varies little over the region, then little or no support is needed. However, if the average is above the critical value, but the local GDP per capita varies greatly, then substantial support will be needed. As noted in section 1, GDP per capita varies greatly between regions in some countries, and much less in others. If the allocation to the region will be based on the average, how confident are we that the poorest part of the region exceeds a minimal critical value $v_{\min}$? Confidence in an allocation can be assessed with the robustness function that assesses the degree of immunity to pernicious uncertainty. Questions similar to those explored in section 3 arise here: how does confidence in the inference vary with the size of the region, with the observed average, and with the critical value?

Uncertainty can be propitious, rather than pernicious, and things can be better that anticipated. That is, uncertainty in the distribution of economic activity may be such that the least productive are in fact quite fruitful. The opportuneness question is: how much uncertainty is needed in order to enable (though not guarantee) an outcome (lowest productivity in the region) that is better than anticipated? This is the converse of the robustness question which is: how much uncertainty can be tolerated in order to guarantee that the outcome (lowest productivity in the region) is acceptable? We derive the robustness function in section 4.2 and the opportuneness function in section 4.3.

The GDP is itself an average over commodities and services. The questions of robustness and opportuneness that we are exploring when considering the spatial average can be similarly addressed to averages over commodities, services or other sectors.

## 4.1 Info-Gap Model of Uncertainty

Let $v(x)$ denote the GDP per capita as a function of location in a region, and let $\overline{v}$ denote the observed average GDP per capita in the region. The policy maker's requirement is that the GDP per capita exceeds a specified minimal value throughout the region:

$$v(x) \geq v_{\min} \quad \text{for all} \quad x \in [0, D] \tag{19}$$

The policy maker wants to know how confident one can be that eq.(19) holds, given the observed regional average, $\overline{v}$.

The GDP per capita as a function of location, $v(x)$, is positive throughout the region, but how much it varies over the region is unknown. What we do know is the average productivity of the region, $\overline{v}$, and limited further information. Let $s$ denote a measure of variation of productivity over the region. We don't know the value of $s$, but we can suppose that $s$ will tend to increase as the size of the region increases. Let $\varepsilon D$ denote a rough estimate of the value of $s$, based perhaps on historical variation. How much $s$ deviates from $\varepsilon D$ is unknown. This uncertainty in $v(x)$ and $s$ can be represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ v(x), s : \; v(x) > 0, \; \left| \frac{v(x) - \overline{v}}{s} \right| \leq h, \; s > 0, \; \left| \frac{s - \varepsilon D}{\varepsilon D} \right| \leq h \right\}, \quad h \geq 0 \tag{20}$$

This info-gap model encodes the observed average, $\overline{v}$, the estimated variation, $\varepsilon D$, and quantifies the uncertain deviation of $v(x)$ and $s$ over the region. Once again, as in the info-gap model of eq.(6) in section 3, the info-gap model is not a single set, but rather an unbounded family of nested sets of $v(x)$ and $s$ values. Furthermore, there is no known worst case and the uncertainty is not probabilistic.

## 4.2 Robustness Against Uncertainty

The robustness to uncertainty in the local GDP per capita and in its variability is the greatest horizon of uncertainty, $h$, up to which the lowest local GDP per capita is no less than the minimal value $v_{\text{min}}$, for all GDP profiles $v(x)$ and variations $s$ in the uncertainty set $\mathcal{U}(h)$:

$$\widehat{h}(v_{\text{min}}) = \max \left\{ h : \; \left( \min_{v,s \in \mathcal{U}(h)} \; \min_{x \in [0, D]} v(x) \right) \geq v_{\text{min}} \right\} \tag{21}$$

Let $m(h)$ denote the inner double minimum in the definition of the robustness function, eq.(21). Define the function $z^+ = z$ for $z \geq 0$ and $z^+ = 0$ otherwise. The minimum on $v(x)$ and $s$ occurs for $v(x) = (\overline{v} - sh)^+$ and $s = (1 + h)\varepsilon D$. This results in a productivity profile that is independent of $x$. Thus we see that:

$$m(h) = [\overline{v} - (1 + h)\varepsilon D h]^+ \tag{22}$$

The robustness is the greatest value of $h$ at which $m(h) \leq v_{\text{min}}$.

Consider values of $h$ that satisfy the relation:

$$(1 + h)\varepsilon D h \leq \overline{v} \tag{23}$$

In that case, the robustness is the solution for $h$ of:

$$\overline{v} - (1 + h)h\varepsilon D = v_{\text{min}} \tag{24}$$

This has one positive root, and it satisfies eq.(23). Thus this root is the robustness:

$$\widehat{h}(v_{\text{min}}) = \frac{-1 + \sqrt{1 + \frac{4}{\varepsilon D}(\overline{v} - v_{\text{min}})}}{2}, \quad \text{for} \quad v_{\text{min}} \leq \overline{v} \tag{25}$$

The robustness function equals zero for values of $v_{\text{min}}$ greater than $\overline{v}$, which means that values greater than the observed average cannot be guaranteed at any horizon of uncertainty.

We note from eq.(25) that the robustness increases as the observed average GDP increases. Also, the robustness decreases as the size of the region, $D$, increases. Specifically:

$$\frac{\partial \widehat{h}(v_{\rm min})}{\partial \overline{v}} \; > \; 0 \tag{26}$$

$$\frac{\partial \widehat{h}(v_{\rm min})}{\partial D} \; < \; 0 \tag{27}$$

As $\overline{v}$, the observed path-average GDP per capita, increases, the info-gap model in eq.(20) shifts to include larger productivity profiles. As a result it becomes less challenging for paths to satisfy the outcome requirement. Eq.(26) expresses this by the robustness increasing as $\overline{v}$ increases. In other words, the robustness, for achieving at least $v_{\rm min}$, increases as the average increases. This is the analog of fig. 4 in which the robustness, for achieving speed no more than $v_{\rm s}$, increases as the speed limit decreases.

As $D$ increases and the reference path gets longer, the requirement that all paths satisfy the outcome requirement at all locations becomes more challenging. This is expressed in eq.(27) by the robustness decreasing as $D$ increases. This is the analog of figs. 1–3 in which the robustness decreases as the distance over which the speed is averaged increases.

## 4.3 Opportuneness From Uncertainty

The opportuneness from uncertainty in the GDP function $v(x)$ and its variability $s$, is the lowest horizon of uncertainty, $h$, at which at least one realization of $v(x)$ and $s$ results in a GDP per capita that is everywhere no less than a wonderfully large value, $v_{\rm max}$. The opportuneness is the complement of the robustness that was defined in eq.(21). Specifically, the opportuneness function is:

$$\widehat{\beta}(v_{\rm max}) = \min\left\{ h : \; \left( \max_{v,s\in\mathcal{U}(h)} \; \min_{x\in[0,\,D]} v(x) \right) \ge v_{\rm max} \right\} \tag{28}$$

Let us compare the definitions of robustness and opportuneness in eqs.(21) and (28). Reading the operators from left to right, the max-min-min of robustness is inverted to min-max-min in the opportuneness. The robustness is the maximum $h$ at which the worst (minimum) couplet $(v, s)$ yields GDP no less than $v_{\rm min}$ at every point. In contrast, the opportuneness is the minimum $h$ at which the best (maximum) couplet $(v, s)$ yields GDP no less than $v_{\rm max}$ at every point. The robustness is the greatest horizon of uncertainty at which the critical outcome is guaranteed, while the opportuneness is the lowest horizon of uncertainty at which the windfall outcome is possible.

Let $M(h)$ denote the inner max-min in eq.(28), which occurs when the GDP function is as large as possible in the uncertainty set $\mathcal{U}(h)$ of the info-gap model in eq.(20). That is, $M(h)$ is obtained from $v(x) = \overline{v} + sh$ and $s = (1+h)\varepsilon D$. Thus:

$$M(h) = \overline{v} + (1+h)h\varepsilon D \tag{29}$$

which is independent of the location, $x$. The opportuneness is the smallest value of $h$ satisfying:

$$\overline{v} + (1+h)h\varepsilon D \ge v_{\rm max} \tag{30}$$

Solving for $h$ in this relation at equality yields the following explicit expression for the opportuneness function:

$$\widehat{\beta}(v_{\max}) = \frac{-1 + \sqrt{1 + \frac{4}{\varepsilon D}(v_{\max} - \overline{v})}}{2}, \quad \text{for} \quad v_{\max} \geq \overline{v} \tag{31}$$

The opportuneness function equals zero for values of $v_{\max}$ lower than $\overline{v}$, which means that no uncertainty is required in order to enable (though not guarantee) values less than the observed average.

We note from eq.(31) that the opportuneness function decreases as the observed average GDP per capita increases. Also, the opportuneness function decreases as the size of the region, $D$, increases. Specifically:

$$\frac{\partial \widehat{\beta}(v_{\max})}{\partial \overline{v}} < 0 \tag{32}$$

$$\frac{\partial \widehat{\beta}(v_{\max})}{\partial D} < 0 \tag{33}$$

Recall that a small value of the opportuneness function indicates that the corresponding outcome is possible (though not guaranteed) at a small horizon of uncertainty, which is desirable. This is distinct from the robustness function for which large values are desirable.

Eq.(32) is readily understood. As the observed average productivity, $\overline{v}$, increases, the info-gap model of eq.(20) shifts to include larger productivity functions $v(x)$. As a result, a productivity function that exceeds the windfall value $v_{\max}$ occurs at a lower horizon of uncertainty. In short, an increase in the average regional productivity enhances the opportuneness.

Eq.(33) is at first surprising. One might not expect that lengthening the reference path enhances the possibility of a productivity profile that exceeds the windfall value $v_{\max}$ at all points on the path. To understand this result we examine the algebraic derivation. We note that $M(h)$ in eq.(29) increases as $D$ increases. That is, the greatest possible value of the minimal productivity, $M(h)$, increases as the path length increases. Algebraically, this results because the range of $s$ increases as the path length, $D$, increases. That is, long paths have greater potential for productivity profiles that are large everywhere. This is precisely what eq.(33) states. It is also true that long paths have the greater potential for productivity profiles that are small everywhere. The opportuneness function expresses a potential for windfall, unlike the robustness function that expresses a guarantee of a critical value.

In comparing eqs.(26) and (32) we see that robustness and opportuneness are *sympathetic* with respect to change in the magnitude of the observed average GDP per capita. (Recall that large $\widehat{h}$ and small $\widehat{\beta}$ are desirable.) A change in $\overline{v}$ that improves one of these functions also causes the other function to improve.

Comparing eqs.(27) and (33) we see that robustness and opportuneness are *antagonistic* with respect to change in the size of the region. A change in size that improves one of these functions causes the other function to become worse.

# 5 Example: Statistical Inference

We first formulate a simple statistical test, and then we introduce info-gap uncertainty and explore robustness and opportuneness.

Consider a process, $Q$, with known mean $\mu_0$ and known variance $\sigma_0^2$. Let $\overline{v}$ be the sample mean of an unknown process. We wish to determine if this mean is consistent with the process $Q$. That is, we wish to decide between the following two hypotheses on the basis of the observed sample mean.

$$\begin{aligned} H_0 &: \quad \text{The unknown process is } Q \\ H_1 &: \quad H_0 \text{ is false} \end{aligned} \tag{34}$$

Consider the statistic:

$$z = \frac{\overline{v} - \mu_0}{\sigma_0} \tag{35}$$

$z$ has a standard normal distribution if $\overline{v}$ is based on a large sample and if $H_0$ is true. Let $\Phi(z)$ denote the standard normal cumulative distribution function. The probability of falsely rejecting $H_0$ is $\alpha$ where:

$$\Phi(|z|) = 1 - \frac{\alpha}{2} \tag{36}$$

$\alpha$ is the level of significance for rejecting $H_0$. That is, for a given value of $z$, a small value of $\alpha$ supports the rejection of $H_0$.

The observation, $\overline{v}$, is the average of measurements of a function $v(x)$ defined on the domain $[0, D]$. The measurements are made at equally spaced points:

$$x_n = (n-1)\delta, \quad n = 1, 2, \ldots, N \tag{37}$$

where the number of measurements, $N$, satisfies:

$$D - \delta < (N-1)\delta \leq D \tag{38}$$

We stress that the measurement increment, $\delta$, is fixed and does not depend on the size of the domain, $D$.

## 5.1 Info-Gap Model of Uncertainty

The sampled function derives from an uncertain process on the domain $[0, D]$. We know several things about that process. The observed average of one realization of the process is known, and we denote it as $\overline{v}_{\text{o}}$. The increment of measurement locations, $\delta$, and the size of the domain, $D$, are known. An estimate of the magnitude of variability of the process, that may depend on the size of the domain, is the known function $s(D)$, though larger variation is definitely possible. We will also adopt the simplifying assumption that the functions $v(x)$ of the unknown process are positive.

Combining this information we can say that realizations of the uncertain process should tend to be around the observed value $\overline{v}_{\text{o}}$ with variation of $\pm s(D)$ or more. We represent this uncertainty with the following info-gap model:

$$\mathcal{U}(h) = \left\{ v(x) : \ v(x) > 0, \ \left| \frac{v(x) - \overline{v}_{\rm o}}{s(D)} \right| \le h \right\}, \quad h \ge 0 \tag{39}$$

Like all info-gap models, the uncertainty is encoded non-probabilistically as an unbounded family of nested sets of possible realizations of the uncertain entity.

The observed sample average, $\overline{v}_{\rm o}$, of a specific realization of the uncertain process generates a value of the statistic $z$ in eq.(35). For simplicity we will assume that:

$$\overline{v}_{\rm o} \ge \mu_0 \tag{40}$$

Given an observed average, $\overline{v}_{\rm o}$, the level of significance is:

$$\widetilde{\alpha} = 2 \left[ 1 - \Phi \left( \frac{\overline{v}_{\rm o} - \mu_0}{\sigma_0} \right) \right] \tag{41}$$

We note that $\widetilde{\alpha}$ increases as $\mu_0$ increases, recalling eq.(40). That is, it is "more difficult" to reject $H_0$ as $\mu_0$, the average of the process $Q$, approaches the observed average, $\overline{v}_{\rm o}$.

## 5.2 Robustness Against Uncertainty

Suppose that this observed level of significance is large enough that $H_0$ would not be rejected. How confident are we in this decision, in light of the uncertainty in the unknown process as represented by the info-gap model of eq.(39)? The level of significance addresses the *statistical* uncertainty of the sampling process, while we must also account for the *info-gap* uncertainty of the unknown process itself. More generally, for any value of $\alpha$ that would lead to acceptance of $H_0$, how confident are we that other realizations of the process would be consistent with this? That is, for any given $\alpha$, and in light of the info-gap uncertainty, how confident are we in accepting $H_0$, as expressed by:

$$\Phi(|z|) \le 1 - \frac{\alpha}{2} \tag{42}$$

We assess the confidence with the robustness function, which is the greatest horizon of uncertainty, $h$, up to which $\Phi(|z|)$ does not exceed the righthand size of eq.(42) for any function in the uncertainty set $\mathcal{U}(h)$. The robustness is defined as follows:

$$\widehat{h}(\alpha) = \max \left\{ h : \ \left( \max_{v \in \mathcal{U}(h)} \Phi(|z(v)|) \right) \le 1 - \frac{\alpha}{2} \right\} \tag{43}$$

A large value of robustness implies high confidence in non-exceedance of the righthand side of eq.(36). Low robustness implies the contrary.

We note that a different robustness function would result if the observed level of significance was small enough to imply rejection of $H_0$. Specifically, the inequalities in eqs.(42) and (43) would be reversed, and the inner maximum in eq.(43) would be a minimum. We will not pursue this further.

The average of a process function $v(x)$ is:

$$\overline{v} = \frac{1}{N} \sum_{n=1}^{N} v(x_n) \tag{44}$$

Let $m(h)$ denote the inner maximum in the definition of the robustness function, eq.(43). $\Phi(\cdot)$ is a monotonically increasing function, and we are assuming eq.(40), so $m(h)$ occurs when $z$ is maximal. From eqs.(35) and (44), this occurs, at horizon of uncertainty $h$, when $v(x) = \overline{v}_{\rm o} + s(D)h$. This maximum does not depend on position. Thus this expression equals the maximum average at horizon of uncertainty $h$ and we can write:

$$m(h) \;=\; \Phi\left(\frac{\overline{v}_{\rm o} + s(D)h - \mu_0}{\sigma_0}\right) \tag{45}$$

The robustness is the greatest horizon of uncertainty, $h$, at which this expression does not exceed $1 - \frac{\alpha}{2}$. That is:

$$\frac{\overline{v}_{\rm o} + s(D)h - \mu_0}{\sigma_0} \leq \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \tag{46}$$

The robustness function is the solution for $h$ of this relation at equality:

$$\widehat{h}(\alpha) = \frac{\sigma_0 \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) - (\overline{v}_{\rm o} - \mu_0)}{s(D)} \tag{47}$$

or zero if this expression is negative.

We see in eq.(47) that the robustness equals zero when $\alpha$ equals the estimated value based on the observed mean, $\widetilde{\alpha}$ in eq.(41).

Eq.(47) also shows that the robustness decreases as $\alpha$ increases from zero to one. A low value of $\alpha$ implies a small level of significance for accepting $H_0$. That is, small $\alpha$ militates against accepting $H_0$. Large robustness implies confidence, vis à vis the info-gap uncertainty in the unknown process, in the corresponding value of $\alpha$. In other words, if the observed level of significance, $\widetilde{\alpha}$, implies acceptance of $H_0$, then the confidence in this acceptance, with respect to the info-gap uncertainty in the function $v(x)$, increases as we consider smaller values of $\alpha$.

We also see from eq.(47) that:

$$\frac{\partial \widehat{h}(\alpha)}{\partial D} = -c \frac{\partial s(D)}{\partial D} \tag{48}$$

where $c$ is a known positive quantity. We recall from the discussion preceding eq.(39) that $s(D)$ is an estimate of the magnitude of variability of the process. This estimate may increase or decrease with the domain size, $D$, or may be independent of $D$.

We note in eq.(47) that the robustness for accepting $H_0$ increases as $\mu_0$ increases. In light of eq.(40), this is because the disparity between $v(x)$ and $Q$ decreases as the disparity decreases between the observed average, $\overline{v}_{\rm o}$, and the average of the process $Q$, which equals $\mu_0$.

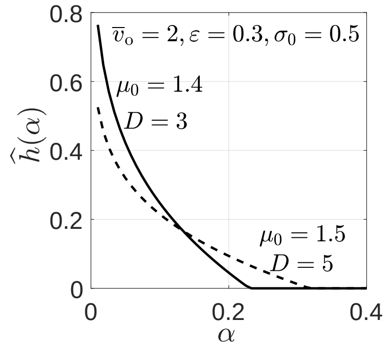


Figure 6: Robustness vs level of significance. $\overline{v}_o = 2$ , $\varepsilon = 0.3$ where $s(D) = \varepsilon D$, $\sigma_0 = 0.5$.

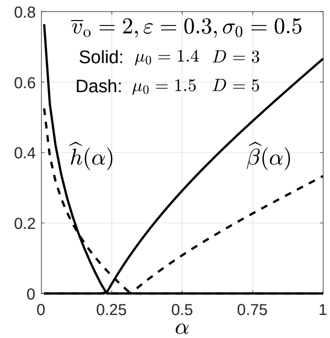


Figure 7: Robustness and opportuneness vs level of significance. $\overline{v}_o = 2$ , $\varepsilon = 0.3$ where $s(D) = \varepsilon D$, $\sigma_0 = 0.7$.

We also see from eq.(47) that:

$$\frac{\partial \widehat{h}(\alpha)}{\partial \alpha} = -\frac{\sigma_0}{2s(D)} \frac{\partial \Phi^{-1}(p)}{\partial p} \le 0 \tag{49}$$

Combining eqs.(48) and (49) we see that the robustness changes with respect to $D$ and $\alpha$ in the same direction if and only if the estimated error, $s(D)$, increases as the domain size, $D$, increases. In that case, an increase in $D$ and a decrease in $\alpha$ result in conflicting trends in the robustness. We can expect the resulting robustness curves to cross one another, as seen previously in fig. 5.

Fig. 6 shows crossing robustness curves for two different choices of $\mu_0$ and $D$, with an attendant dilemma for the decision maker in planning the measurement and in accepting $H_0$. The estimated level of significance, $\widetilde{\alpha}$ in eq.(41), increases as $\mu_0$ increases, and does not depend on $D$. However, the robustness in eq.(47) decreases as the domain size, $D$, increases. More precisely, the slope of the robustness curve decreases as $D$ increases, implying that $\widehat{h}$ increases more slowly as $\alpha$ decreases. Succinctly, the "cost of robustness" increases as $D$ increases. Thus the combination $(\mu_0, D) = (1.5, 5)$ is putatively more definitive (larger $\widetilde{\alpha}$) than $(\mu_0, D) = (1.4, 3)$. However, the cost of robustness is greater for $(1.5, 5)$ than for $(1.4, 3)$. The result is that the robustness curves cross one another, and neither option is robust dominant. Stated differently, the choice of $\mu_0 = 1.5$ gives greater confidence in accepting $H_0$, if there is no info-gap uncertainty. However, in light of that uncertainty, one may choose to forego the larger domain and to accept the lower $\mu_0$ in order to enhance the robustness to uncertainty.

## 5.3 Opportuneness From Uncertainty

The robustness function protects against pernicious uncertainty, and the opportuneness function addresses the possibility of wonderful outcomes resulting from propitious uncertainty. Continuing as in section 5.2, we suppose that the observed level of significance, $\widetilde{\alpha}$ in eq.(41), is large enough so that $H_0$ would not be rejected. A larger value of $\alpha$ would imply even greater statistical confidence in accepting $H_0$. How much info-gap uncertainty in the function $v(x)$ is needed for such wonderfully strong acceptance of $H_0$ to be possible?

Note that eq.(42) implies that $\alpha \leq 2\left[1 - \Phi(|z|)\right]$. Thus a small value of $\Phi(|z|)$ implies a large value of $\alpha$. Hence we ask: what is the lowest horizon of uncertainty, $h$, at which $\Phi(|z|)$ could be quite small and $\alpha$ could be quite large? In other words, for any value of $\alpha$, what is the lowest horizon of uncertainty at which the lowest value of $\Phi(|z|)$ is not greater than $1 - \frac{\alpha}{2}$? The answer is the opportuneness function whose formal definition is:

$$\widehat{\beta}(\alpha) = \min \left\{ h : \left( \min_{v \in \mathcal{U}(h)} \Phi(|z(v)|) \right) \leq 1 - \frac{\alpha}{2} \right\} \tag{50}$$

If the horizon of uncertainty is no less than $\widehat{\beta}(\alpha)$ then it is possible, though not guaranteed, that the level of significance will be as large as $\alpha$.

Comparing the opportuneness function in eq.(50) with the robustness function in eq.(43) we see that they are complementary: the min-min of opportuneness is the complement of the max-max of robustness. Their meanings are also complementary. The robustness is the greatest horizon of uncertainty at which failure cannot occur, while the opportuneness is the lowest horizon of uncertainty at which wonderful windfall is possible.

We derive an explicit expression for the opportuneness function as follows, continuing with the assumption of eq.(40).

Let $M(h)$ denote the inner minimum in the definition of the opportuneness function in eq.(50). $\Phi(|z(v)|)$ decreases monotonically as $|z|$ decreases. Thus this inner minimum occurs when $|z|$ is as small as possible. Hence, since $v(x) > 0$, this occurs when $v(x)$ is minimal at each measurement point. Thus, in analogy to eq.(45), and recalling the assumption of eq.(40), the inner minimum is:

$$M(h) = \Phi\left( \frac{(\overline{v}_{\rm o} - s(D)h)^{+} - \mu_0}{\sigma_0} \right) \tag{51}$$

(Recall the definition of an exponent "+" following eq.(21).) From eq.(41) we see that the righthand side equals $1 - \frac{\widetilde{\alpha}}{2}$ when $h = 0$. At larger $h$ the righthand side is lower and equals $1 - \frac{\alpha}{2}$ for some $\alpha > \widetilde{\alpha}$. Equating $M(h)$ to $1 - \frac{\alpha}{2}$ and solving for $h$ yields the opportuneness as a function of the level of significance $\alpha$:

$$\widehat{\beta}(\alpha) = \frac{\overline{v}_{\rm o} - \mu_0 - \sigma_0 \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)}{s(D)} \tag{52}$$

or zero if this is negative. $\widehat{\beta}(\alpha)$ increases from zero as $\alpha$ increases from $\widetilde{\alpha}$. This expresses the opportuneness trade off: a greater value of the level of significance, implying stronger acceptance of $H_0$, is possible only at greater horizon of uncertainty.

Fig. 7 shows opportuneness curves for two choices of the parameters $\mu_0$ and $D$. The robustness curves with these parameter values are reproduced from fig. 6. The positive slope of each opportuneness curve expresses a trade off whose meaning is that a large and desirable level of significance, $\alpha$, is possible — though not guaranteed — at a large horizon of uncertainty.

Employing eqs.(47) and (52) we see that the robustness and opportuneness depend on the domain size, $D$, only through the uncertainty weight $s(D)$. Let us suppose, then, that $s(D)$ in fact depends on $D$. We now see that, for any values of $\alpha_{\rm c}$ and $\alpha_{\rm w}$:

$$\frac{\partial \widehat{h}(\alpha_{\rm c})}{\partial D}\frac{\partial \widehat{\beta}(\alpha_{\rm w})}{\partial D} = \frac{\partial \widehat{h}(\alpha_{\rm c})}{\partial s}\frac{\partial \widehat{\beta}(\alpha_{\rm w})}{\partial s}\left(\frac{\partial s}{\partial D}\right)^2 > 0 \tag{53}$$

Any change in the size of the domain, $D$, that increases the robustness function, $\widehat{h}(\alpha_{\rm c})$, also increases the opportuneness function, $\widehat{\beta}(\alpha_{\rm w})$, at any values of $\alpha_{\rm c}$ and $\alpha_{\rm w}$. The robustness function is the immunity against failure so a large value is desirable, while the opportuneness function is the immunity against wonderful windfall so a small value is desirable. Eq.(53) shows that robustness and opportuneness are antagonistic with respect to the size of the domain: Any change in $D$ that improves one of these functions worsens the other.

In contrast, eqs.(47) and (52) show that robustness and opportuneness are sympathetic with respect to change in $\overline{v}_{\rm o}$, $\mu_0$ or $\sigma_0$:

$$\frac{\partial \widehat{h}(\alpha_{\rm c})}{\partial x}\frac{\partial \widehat{\beta}(\alpha_{\rm w})}{\partial x} \le 0 \tag{54}$$

when $x$ is $\overline{v}_{\rm o}$, $\mu_0$ or $\sigma_0$. Any change in one of these variables that improves one of these functions also improves the other.

# 6 Robustness as a Proxy for the Probability of Success

The robustness to uncertainty for achieving a specified goal, as defined in eq.(4) and based on an info-gap model of uncertainty satisfying eqs.(2) and (3), is entirely non-probabilistic. Nonetheless, in some situations a change in the planner's decision that augments the robustness will also augment the probability of success. When this is true we say that robustness is a *proxy* for the probability of achieving the goal. When the proxy property holds one can enhance or maximize the probability of success even when one has no knowledge of the relevant probability distributions. One will not know the value of the probability of success, but by maximizing the non-probabilistic robustness to uncertainty one will also maximize the probability of success, when the proxy property holds. Robustness is not always a proxy for probability, though it arises in a range of situations, including management of risky assets, forecasting, and foraging by animals (Ben-Haim, 2009, 2014). However, Davidovitch (2009) has shown that very strict conditions must prevail in order for the proxy property to hold.

In this section we explore the proxy property for the examples in the previous sections. In sections 6.1, 6.2 and 6.3 we identify specific conditions under which the proxy property holds for the speed limit example of section 3, the economic example of section 4, and the statistical inference of section 5.

## 6.1 Speed Limits

We begin by demonstrating that the robustness increases as the speed limit, $v_\ell$, increases. We then explore the dependence of the probability of success on the speed limit. Combining these considerations will yield the proxy property.

The conditions on the speed-profile function, $v(t)$, in the info-gap model of eq.(6) at horizon of uncertainty $h$, imply that:

$$0 < v(t) \le (1+h)v_\ell \quad \text{for all} \quad t \ge 0 \tag{55}$$

Consider the info-gap model of eq.(6) for two different values of the speed limit, $\mathcal{U}(h, v_\ell)$ and $\mathcal{U}(h, v'_\ell)$. Eq.(55) implies that the uncertainty sets are nested as $v_\ell$ increases:

$$v_\ell < v'_\ell \quad \text{if and only if} \quad \mathcal{U}(h, v_\ell) \subseteq \mathcal{U}(h, v'_\ell) \tag{56}$$

Recalling the definition of $m(h,t)$ following eq.(7) we see that eq.(56) implies:

$$v_\ell < v'_\ell \quad \text{if and only if} \quad m(h,t|v_\ell) \le m(h,t|v'_\ell) \tag{57}$$

Recalling that $m(h,t)$ is the inverse of the robustness defined in eq.(7), we see that eq.(57) implies:

$$v_\ell < v'_\ell \quad \text{if and only if} \quad \widehat{h}(v_\mathrm{s}, v_\ell) \ge \widehat{h}(v_\mathrm{s}, v'_\ell) \tag{58}$$

Robustness increases if and only if the speed limit decreases.

We now consider the probability of success, which is the probability that the maximum velocity does not exceed the safe speed $v_\mathrm{s}$. For notational convenience let $\mu$ denote the maximum velocity during $[0, T]$:

$$\mu = \max_{t\in[0,T]} v(t) \tag{59}$$

The formal definition of the probability of success is the following cumulative probability distribution function:

$$P_\mathrm{s}(v_\mathrm{s}|v_\ell) = \mathrm{Prob}\,(\mu \le v_\mathrm{s}|\ v_\ell) \tag{60}$$

Let $p_\mathrm{s}(v_\mathrm{s}|v_\ell)$ denote the corresponding probability density function of $\mu$. We can define these probability functions, but we cannot calculate them because they are unknown.

Recall that $v_\ell$ is the regulator's choice of the speed limit, while $v_\mathrm{s}$ is the greatest safe speed that depends on the road and driving conditions. One might reasonably anticipate that $p_\mathrm{s}(v_\mathrm{s}|v_\ell)$ — viewed as a function of $v_\mathrm{s}$ — shifts to higher values as $v_\ell$ is raised, reflecting drivers' lowered restraint as the regulator raises the speed limit. That is, one might anticipate that higher speed limits are imposed when higher speeds are safe. However, there is no apodictic necessity for this shift in $p_\mathrm{s}(v_\mathrm{s}|v_\ell)$. One concludes that, *if* $p_\mathrm{s}(v_\mathrm{s}|v_\ell)$ shifts to the higher values as $v_\ell$ is raised, *then:*

$$v_\ell < v'_\ell \quad \text{if and only if} \quad P_\mathrm{s}(v_\mathrm{s}|v_\ell) \ge P_\mathrm{s}(v_\mathrm{s}|v'_\ell) \tag{61}$$

Continuing with this assumption regarding $p_\mathrm{s}(v_\mathrm{s}|v_\ell)$, we see from eqs.(58) and (61) that robustness is a proxy for probability of success:

$$\widehat{h}(v_\mathrm{s}, v_\ell) \ge \widehat{h}(v_\mathrm{s}, v'_\ell) \quad \text{if and only if} \quad P_\mathrm{s}(v_\mathrm{s}|v_\ell) \ge P_\mathrm{s}(v_\mathrm{s}|v'_\ell) \tag{62}$$

We stress that this proxy property is not universal, and does not necessarily hold if $p_s(v_s|v_\ell)$ does not shift to higher values as $v_\ell$ is raised. We also note that, while eq.(62) is bi-directional and symmetric between robustness and probability, the proxy property refers specifically to the implication from robustness to probability. The implication from robustness to probability is significant because the robustness function is epistemically sparser than the probability distribution and entails no explicit knowledge of probabilities. When the proxy property of eq.(62) holds, one can enhance the probability of success by enhancing the non-probabilistic robustness to uncertainty.

## 6.2 Economic Activity

We now derive a proxy property for the robustness function of economic activity.

The robustness function in eq.(25) immediately shows that robustness increases as the average GDP increases. More precisely:

$$\overline{v} < \overline{v}' \quad \text{if and only if} \quad \widehat{h}(v_{\min}|\overline{v}) \leq \widehat{h}(v_{\min}|\overline{v}') \tag{63}$$

We now define the probability of success, which is the probability that the minimum GDP along the path is no less than $v_{\min}$, conditioned on the average GDP, $\overline{v}$. Define the minimum along the path:

$$\mu = \min_{x \in [0,D]} v(x) \tag{64}$$

The probability of success is defined as:

$$P_s(v_{\min}|\overline{v}) = \text{Prob}(\mu \geq v_{\min}|\overline{v}) \tag{65}$$

Let $p_s(v_{\min}|\overline{v})$ denote the probability density function for $P_s(v_{\min}|\overline{v})$.

One can reasonably expect that the distribution of minimal GDP values shifts to the right as the average GDP increases, recognizing that this does not necessarily hold. That is, one can expect that $p_s(v_{\min}|\overline{v})$, viewed as a function of $v_{\min}$, shifts to the right as $\overline{v}$ increases. We refer to this as the "shifting property", which asserts:

$$\overline{v} < \overline{v}' \quad \text{if and only if} \quad P_s(v_{\min}|\overline{v}) \leq P_s(v_{\min}|\overline{v}') \tag{66}$$

When this shifting property holds we see that eqs.(63) and (66) imply that robustness is a proxy for probability of success:

$$\widehat{h}(v_{\min}|\overline{v}) \leq \widehat{h}(v_{\min}|\overline{v}') \quad \text{if and only if} \quad P_s(v_{\min}|\overline{v}) \leq P_s(v_{\min}|\overline{v}') \tag{67}$$

## 6.3 Statistical Inference

We now consider the proxy property for the robustness function in eq.(47), which presumes that the observed level of significance, eq.(41), is large enough that $H_0$ would not be rejected.

We know the mean of the process $Q$, and we consider two different realizations of this mean:

$$\mu_0 < \mu_0' \tag{68}$$

where both values satisfy eq.(40) for the observed average value, $\overline{v}_o$.

The robustness function in eq.(47) shows that the robustness for not rejecting $H_0$ increases as the mean of the process $Q$ increases:

$$\mu_0 < \mu_0' \quad \text{if and only if} \quad \widehat{h}(\alpha|\mu_0) \le \widehat{h}(\alpha|\mu_0') \tag{69}$$

This means that the robustness for not rejecting $H_0$ increases as the average of the process $Q$ increases.

Now consider the probability of success, which we define as the probability that $|z|$ does not exceed the righthand size of eq.(42), given that $H_0$ holds:

$$P_s(\alpha|\mu_0) = \text{Prob}\left(\Phi(|z|) \le 1 - \frac{\alpha}{2}\right) \tag{70}$$

where $z$ is defined in eq.(35).

It is evident that:

$$\mu_0 < \mu_0' \quad \text{if and only if} \quad |z(\mu_0)| > |z(\mu_0')| \quad \text{if and only if} \quad \Phi(|z(\mu_0)|) \ge \Phi(|z(\mu_0')|) \tag{71}$$

where both $\mu$ and $\mu'$ satisfy eq.(40). Thus the probability of success increases as the mean of the process $Q$ increases. That is, combining eqs.(70) and (71) we see that:

$$\mu_0 < \mu_0' \quad \text{if and only if} \quad P_s(\alpha|\mu_0) \le P_s(\alpha|\mu_0') \tag{72}$$

Combining eqs.(69) and (72) we see that

$$\widehat{h}(\alpha|\mu_0) \le \widehat{h}(\alpha|\mu_0') \quad \text{if and only if} \quad P_s(\alpha|\mu_0) \le P_s(\alpha|\mu_0') \tag{73}$$

This means that the robustness is a proxy for the probability of success.

# 7 Conclusion

We have studied three examples with the concepts of robustness and opportuneness as developed in info-gap decision theory. The robustness function equals the greatest horizon of uncertainty up to which a critical requirement is guaranteed to be satisfied. This underlies the robust-satisficing prioritization of decision alternatives: one option is preferred over another option if the first option satisfies the requirement over a greater range of uncertainty than the second option. What is optimized is the robustness to uncertainty, while outcomes are satisficed rather than optimized. The opportuneness function equals the lowest horizon of uncertainty at which wonderful windfall outcomes — better than anticipated — are possible, though not necessarily guaranteed. This underlies the opportune-windfalling prioritization of options.

Section 3 discussed the choice of speed limits to deter dangerous driving, where the speed profile of the driver is deeply uncertain. We observed that the predicted velocity has no robustness to uncertainty in the speed profile (this illustrates the zeroing property). Only velocities in excess of the prediction have positive robustness (this illustrates the trade off property). We examined two decision variables: the speed limit and the size of the domain within which the speed is measured. We showed that robustness to uncertainty decreases as either the speed limit or the domain-size increases. Simultaneous changes of these two factors in different directions therefore have conflicting impact on robustness to uncertainty. This can cause the corresponding robustness curves to cross one another, and this raises the potential for a reversal of preference between different combinations of these decision variables.

Section 4 explored inference about extremes of economic activity based on an observed spatial average of the GDP, with implications for policy to ameliorate poverty. We considered both pernicious and propitious uncertainty in the spatial variation of the GDP per capita. We found that the robustness increases as the observed path-averaged GDP per capita increases. However, the robustness decreases as the path-length for averaging the GDP increases. In contrast, the opportuneness from propitious uncertainty improves both as the path length grows and as the observed average GDP grows.

Section 5 explored a statistical hypothesis test regarding an unknown variable process whose mean has been measured. We showed that the robustness decreases as the level of significance, $\alpha$, increases. In other words, if the observed level of significance implies acceptance of the null hypothesis, then the confidence in this acceptance, with respect to the info-gap uncertainty, increases as we consider smaller values of the level of significance. We demonstrated the potential for reversal of preference between different choices of the planning variables, as expressed by crossing of the corresponding robustness curves. We also explored the opportuneness from propitious uncertainty. We showed that any change in the domain size that improves the robustness causes deterioration in the opportuneness: these two functions are antagonistic with respect to the size of the domain on which the uncertain function is defined. In contrast, we showed that robustness and opportuneness are sympathetic with respect to change in any of several other design variables.

Section 6 discussed the concept of robustness as a proxy for the probability of success. When the proxy property holds, the probability of success is maximized by maximizing the robustness to uncertainty, even though the latter function entails no probabilistic information. We demonstrated specific conditions under which the robustness functions of sections 3 to 5 display the proxy property.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

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# A Maximum Acceleration

Consider any infinitesimal segment of length $\mathrm{d}x$ along the road. The speeds at the start and end of this segment are related as:

$$v(x+\mathrm{d}x) = v(x) + \frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x \tag{74}$$

The derivative in this relation can be written:

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} = \frac{\mathrm{d}v(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\dot{v}(t)}{v(x)} \tag{75}$$

where $\dot{v}(t)$ is the temporal acceleration of the car. Combining the last two relations yields:

$$v(x+\mathrm{d}x) = v(x) + \frac{\dot{v}(t)}{v(x)}\mathrm{d}x \tag{76}$$

Hence, assuming that $v(x)$ is positive, the increment in velocity on any infinitesimal segment of road is maximal if the car accelerates maximally. The cumulative effect is that the final speed is maximal if the car accelerates maximally throughout the travel, even though this actually minimizes the time during which acceleration occurs.