Lecture Notes on Robust-Satisficing Behavior

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Source material: Yakov Ben-Haim, 2001, 2nd edition, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press. Chapter 11.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

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1 Introduction

¶ We have considered 3 decision paradigms in this course:

- Optimizing:

$$q^\star = \arg\max_q R(q, \widetilde{u}) \tag{1}$$

- Robust satisficing:

$$q_{\rm rs} = \arg\max_q \widehat{h}(q, R_{\rm c}) \tag{2}$$

- Opportune windfalling:

$$q_{\rm ow} = \arg\min_q \widehat{\beta}(q, R_{\rm c}) \tag{3}$$

¶ Do we observe satisficing in practice?

- Yes.
- Why?

¶ We will consider info-gap robust-satisficing explanations of classical paradoxes of behavior under uncertainty:

- Ellsberg (section 2).
- Allais (section 3).

¶ We will study an info-gap robust-satisficing approach to expected-utility risk aversion (section 4).

¶ We will study an info-gap robust-satisficing approach to foraging by animals in section 5.

¶ We will show why info-gap robust-satisficing is a better bet than direct optimization in a wide class of situations (section 6).

2 The Ellsberg Paradox

¶ References:

- Yakov Ben-Haim, 2001, 2nd edition, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press. Section 11.1.
- Mas-Colell, A., M.D. Whinston and J.R. Green, 1995, *Microeconomic Theory,* Oxford University Press, p.207.
- Ellsberg, D., 1961, Risk, ambiguity and the Savage axioms, *Quarterly Journal of Economics,* 75: 643–669.

¶ Ellsberg's urn experiment:

- R contains 49 white and 51 black balls, well mixed.
- H contains 100 white and black balls.
- First experiment: choose one ball and win $1000 if it is black.
  Students: from which urn would you choose the ball?
- Second experiment: choose one ball and win $1000 if it is white.
  Students: from which urn would you choose the ball?

¶ Ellsberg's observations:

- 1st experiment: most agents choose R-ball.
- 2nd experiment: most agents choose R-ball again.

¶ Ellsberg's paradox:

- R-urn probabilities of W and B: 0.49 and 0.51.
- H-urn probabilities of W and B: π and $1-\pi$.
- 1st experiment implies $1-\pi < 0.51$ hence $\pi > 0.49$.
- Hence agent should choose H-ball in 2nd experiment. Most don't.

2.1 Info-Gap Analysis of 1st Experiment

¶ In the 1st experiment the agent must choose between the following two lotteries:

$$\begin{array}{lcccccc} & & & & \text{\$1000} & & \text{\$0} \\ & & & & \text{(black)} & & \text{(white)} \\ \text{(Choose } R\text{-ball)} & L_1 : & \widetilde{p}_1 & = & (0.51 & , & 0.49) \\ \text{(Choose } H\text{-ball)} & L_1' : & \widetilde{p}_1' & = & (1-\pi & , & \pi) \end{array} \quad (4)$$

¶ Uncertainty:

- Agent does not perform precise probability calculations.
- Subjective perception of probabilities is unclear. E.g.:
  - What is precise meaning of 0.02 probability difference between $\widetilde{p}_{11}$ and $\widetilde{p}_{12}$?
  - What is precise meaning of π?
- Info-gap models:

$$\mathcal{L}(h, \widetilde{p}_1) = \left\{ p_1 \in \mathcal{P}_2 : \left| \frac{p_{11} - \widetilde{p}_{11}}{\widetilde{p}_{11}} \right| \le h \right\}, \quad h \ge 0 \quad (5)$$

$$\mathcal{L}'(h, \widetilde{p}_1') = \left\{ p_1 \in \mathcal{P}_2 : \left| \frac{p_{11} - \widetilde{p}_{11}'}{\widetilde{p}_{11}'} \right| \le \eta h \right\}, \quad h \ge 0 \quad (6)$$

$\eta > 1$ implies that $\mathcal{L}'(h, \widetilde{p}_1')$ is an expanded and shifted version of $\mathcal{L}(h, \widetilde{p}_1)$.

¶ Known utility of winning \$1000 is $u > 0$. Utility of winning \$0 is 0.

¶ Decision variable:

$$q = \begin{cases} 1 & \text{for choosing } L_1 \\ -1 & \text{for choosing } L_1' \end{cases} \quad (7)$$

¶ Robustness function for satisficing expected utility:

$$\widehat{h}(q, r_c) = \max \left\{ h : \underbrace{\left(\frac{1}{2}|1+q| \min_{p \in \mathcal{L}(h, \widetilde{p}_1)} u p_{11} + \frac{1}{2}|1-q| \min_{p \in \mathcal{L}'(h, \widetilde{p}_1')} u p_{11} \right)}_{\mathcal{R}_*[q, \mathcal{L}(h, \widetilde{p}_1), \mathcal{L}'(h, \widetilde{p}_1')]} \ge r_c \right\} \quad (8)$$

¶ Evaluating robustness.

- $\mu_1(h)$ and $\mu_1'(h)$ denote the first and second inner minima in eq.(8).
- $\mu_1(h)$ and $\mu_1'(h)$ decrease monotonically as h increases.
- When $q = 1$, robustness is max h at which $\mu_1(h) = r_c$.

  That is, $\mu_1(h)$ is the inverse of $\widehat{h}(1, r_c)$:

$$\mu_1(h) = r_c \quad \text{if and only if} \quad \widehat{h}(1, r_c) = h \tag{9}$$

  Plot of $\mu_1(h)$ vs. h is the same as a plot of r_c vs. $\widehat{h}(1, r_c)$.
- When $q = -1$, a plot of $\mu_1'(h)$ vs. h is the same as a plot of r_c vs. $\widehat{h}(-1, r_c)$:

$$\mu_1'(h) = r_c \quad \text{if and only if} \quad \widehat{h}(-1, r_c) = h \tag{10}$$

- One readily finds the following expressions for these μ-functions:

$$\mu_1(h) = \max[0,\ (1-h)u\widetilde{p}_{11}] \tag{11}$$
$$\mu_1'(h) = \max[0,\ (1-\eta h)u\widetilde{p}_{11}'] \tag{12}$$

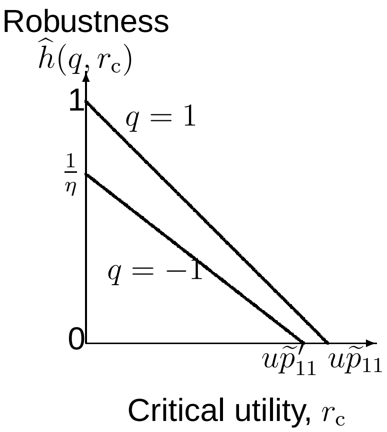

Figure 1: Robustness curves for the **first pair of lotteries,** eq.(4).

¶ Robust-satisficing choice in 1st experiment.

- Robustness curves based on eqs.(9)–(12) are shown in fig. 1.
- Robustness of $q = 1$ is greater than the robustness of lottery L_1' for any attainable r_c.
- Because:
  - ○ $\widetilde{p}_{11} = 0.51 > \widetilde{p}_{11}' = 1 - \pi$ (recall that π evidently exceeds 0.49).
  - ○ $\eta > 1$.
- The robust-satisficing agent will choose lottery L_1 over L_1'.
- The utility-optimizing agent will choose lottery L_1 over L_1'.
- They agree on the action, but for different reasons.

2.2 Info-Gap Analysis of 2nd Experiment

¶ In the 2nd experiment the agent must choose between the following two lotteries:

$$\begin{array}{lllllll} & & & & \text{\$1000} & & \text{\$0} \\ & & & & \text{(white)} & & \text{(black)} \\ \text{(Choose } R\text{-ball)} & L_2: & \widetilde{p}_2 & = & (0.49 & , & 0.51) \\ \text{(Choose } H\text{-ball)} & L_2': & \widetilde{p}_2' & = & (\pi & , & 1-\pi) \end{array} \quad (13)$$

From 1st experiment it would seem that $\pi > 0.49$.

¶ Uncertainties:

- Perceived probabilities are again uncertain.
- Info-gap models $\mathcal{L}(h, \widetilde{p}_2)$ and $\mathcal{L}'(h, \widetilde{p}_2')$ in eqs.(5) and (6) on p.4.
- New center points $\widetilde{p}_2$ and $\widetilde{p}_2'$.
- $\eta > 1$ implies that $\mathcal{L}'(h, \widetilde{p}_2')$ is an expanded and shifted version of $\mathcal{L}(h, \widetilde{p}_2)$.

¶ μ-functions for robustness:

$$\mu_2(h) = \max[0,\ (1-h)u\widetilde{p}_{21}] \quad (14)$$

$$\mu_2'(h) = \max[0,\ (1-\eta h)u\widetilde{p}_{21}'] \quad (15)$$

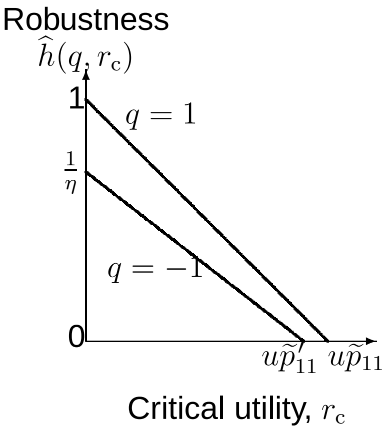

Figure 2: Robustness curves for the **first pair of lotteries,** eq.(4).

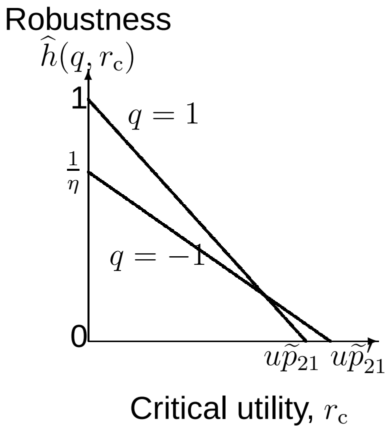

Figure 3: Robustness curves for the **second pair of lotteries,** eq.(13).

¶ Robust-satisficing choice in 2nd experiment:

- Robustness curves for lotteries L_2 and L_2' in fig. 3 on p.7.
- These robustness curves cross because:
  - ∘ $\eta > 1$ as before.
  - ∘ $\widetilde{p}_{21}' = \pi > \widetilde{p}_{21} = 0.49$.
- However, the difference between $\widetilde{p}_{21}'$ and $\widetilde{p}_{21}$ is likely to be small so the intersection occurs at very low robustness, especially if η is very large.
- Most decision makers may be expected to require robustness to uncertainty in excess of the robustness at which the curves cross, so the robust-satisficing agent will again tend to choose lottery L_2 over L_2'. This is what is usually observed.
- The utility-optimizing agent will choose L_2' over L_2.

¶ Summary:

- EU-maximization predicts: choose L_2 in 1st experiment, L_2' in 2nd experiment.
- Robust-satisficing predicts: choose L_2 in both experiments.
- Most decision makers behave according to the robust-satisficing model.

3 The Allais Paradox

¶ References:

• Yakov Ben-Haim, 2001, 2nd edition, 2006, *Information-Gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press. Section 11.2.

• Mas-Colell, A., M.D. Whinston and J.R. Green, 1995, *Microeconomic Theory,* Oxford University Press, p.179.

3.1 Allais' Experiment

¶ Lotteries:

• Lottery has 3 prizes with values $V_1 = \$0$, $V_2 = \$0.5 \times 10^6$ and $V_3 = \$2.5 \times 10^6$.

• Lottery is specified by $L = (p_1, p_2, p_3)$ where p_n is the probability of winning V_n.

¶ DM is twice offered a choice between two lotteries.

• The first pair of lotteries between which the DM must choose is:

$$L_1 = (0, 1, 0) \quad \text{and} \quad L_1' = (0.1, 0.89, 0.01) \tag{16}$$

Students: which lottery do you prefer?

• The second pair of lotteries between which the agent chooses is:

$$L_2 = (0, 0.11, 0.89) \quad \text{and} \quad L_2' = (0.1, 0, 0.9) \tag{17}$$

Students: which lottery do you prefer?

¶ Allais' observation.

• Usually, people who prefer L_1 over L_1' also prefer L_2' over L_2:

$$L_1 \succ L_1' \quad \text{implies} \quad L_2' \succ L_2 \tag{18}$$

• This pair of choices is not universal.

• For some people: $L_1 \succ L_1'$ and $L_2 \succ L_2'$:

$$L_1 \succ L_1' \quad \text{implies} \quad L_2 \succ L_2' \tag{19}$$

• Other combinations of preferences are also observed.

¶ The paradox.

- $u = (u_1, u_2, u_3) =$ utility vector for prizes V_1, V_2, V_3.
- Expected utility of a lottery:

$$\text{EU}(L) = u^T L \tag{20}$$

- Note that:

$$L_2 = L_1 + \phi \quad \text{and} \quad L'_2 = L'_1 + \phi \tag{21}$$

where $\phi = (0, -0.89, 0.89)$.

- Hence:

$$\begin{aligned}
\text{EU}(L_1) > \text{EU}(L'_1) &\iff u^T L_1 > u^T L'_1 & (22)\\
&\iff u^T L_1 + u^T \phi > u^T L'_1 + u^T \phi & (23)\\
&\iff u^T L_2 > u^T L'_2 & (24)\\
&\iff \text{EU}(L_2) > \text{EU}(L'_2) & (25)
\end{aligned}$$

- Hence $L_1 \succeq L'_1$ entails $L_2 \succeq L'_2$ if the preferences maximize expected utility.
- Thus, $L_1 \succeq L'_1$ with $L'_2 \succ L_2$, violates additivity axiom of expected-utility theory.

¶ A decision theory must account for:

- Diversity of response to Allais' experiment.
- Common violation of additivity axiom of expected-utility theory.

¶ We will see that robust-satisficing behavior does this by considering two situations:

- Uncertain perceived probabilities of the lotteries leads to preference reversal, section 3.2.
- Uncertain utilities leads to expected-utility maximization, section 3.3.

3.2 Probability Uncertainty

¶ The DM is uncertain only about subjective meaning of the probabilities.

¶ 1st pair of lotteries, eq.(16) on p.8:

- L_1 is perceived more clearly and with less uncertainty than L'_1.
- Fractional-error info-gap model, like eqs.(5) and (6), is:

$$\mathcal{L}_1(h, \widetilde{p}_1) = \left\{ p_1 \in \mathcal{P}_3 : \left| \frac{p_{1i} - \widetilde{p}_{1i}}{\widetilde{p}_{1i}} \right| \leq h, \; i = 2, 3 \right\}, \; h \geq 0 \quad (26)$$

$$\mathcal{L}'_1(h, \widetilde{p}'_1) = \left\{ p_1 \in \mathcal{P}_3 : \left| \frac{p_{1i} - \widetilde{p}'_{1i}}{\widetilde{p}'_{1i}} \right| \leq \eta h, \; i = 2, 3 \right\}, \; h \geq 0 \quad (27)$$

$\eta > 1$ implies that $\mathcal{L}'_1(h, \widetilde{p}'_1)$ is an expanded and shifted version of $\mathcal{L}_1(h, \widetilde{p}_1)$.

¶ 2nd pair of lotteries, eq.(17), p.8:

- L_2 is more intricate and confusing that L'_2.
- The info-gap models are:

$$\mathcal{L}_2(h, \widetilde{p}_2) = \left\{ p_2 \in \mathcal{P}_3 : \left| \frac{p_{2i} - \widetilde{p}_{2i}}{\widetilde{p}_{2i}} \right| \leq \eta h, \; i = 2, 3 \right\}, \; h \geq 0 \quad (28)$$

$$\mathcal{L}'_2(h, \widetilde{p}'_2) = \left\{ p_2 \in \mathcal{P}_3 : \left| \frac{p_{2i} - \widetilde{p}'_{2i}}{\widetilde{p}'_{2i}} \right| \leq h, \; i = 2, 3 \right\}, \; h \geq 0 \quad (29)$$

$\eta > 1$ so now $\mathcal{L}_2(h, \widetilde{p}_2)$ is an expanded and shifted version of $\mathcal{L}'_2(h, \widetilde{p}'_2)$.

¶ Decision variable for choosing between lotteries L_n and L'_n is q:

$$q = \begin{cases} 1 & \text{for choosing } L_n \\ -1 & \text{for choosing } L'_n \end{cases} \quad (30)$$

¶ $\widetilde{u}$ = known personal utility vector.

¶ Robustness function:

$$\widehat{h}_n(q, r_\mathrm{c}) = \max \left\{ h : \underbrace{\left(\frac{1}{2}|1+q| \min_{p \in \mathcal{L}_n(h, \widetilde{p}_n)} \widetilde{u}^T p \; + \; \frac{1}{2}|1-q| \min_{p \in \mathcal{L}'_n(h, \widetilde{p}'_n)} \widetilde{u}^T p \right)}_{\mathcal{R}_*[q, \mathcal{L}_n(h, \widetilde{p}_n), \mathcal{L}'_n(h, \widetilde{p}'_n)]} \geq r_\mathrm{c} \right\} \quad (31)$$

¶ Evaluating the robustness.

• $\mu_n(h)$ and $\mu_n'(h)$ denote the first and second inner minima in eq.(31).

• As explained in eqs.(9) and (10), $\mu_n(h)$ is the inverse of $\widehat{h}_n(1, r_{\rm c})$ while $\mu_n'(h)$ is the inverse of $\widehat{h}_n(-1, r_{\rm c})$.

• Utilities of zero, moderate, and high reward are $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{u}_3$. Assume: $\widetilde{u}_2$ and $\widetilde{u}_3$ are positive and $\widetilde{u}_1 = 0$.

• The μ-functions for the first pair of lotteries, eq.(16), p.8, are obtained by choosing p_{12} and p_{13} as small as possible at horizon of uncertainty h:

$$\mu_1(h) = \max[0,\ (1-h)\widetilde{u}_2\widetilde{p}_{12}] + \max[0,\ (1-h)\widetilde{u}_3\widetilde{p}_{13}] \tag{32}$$

$$\mu_1'(h) = \max[0,\ (1-\eta h)\widetilde{u}_2\widetilde{p}_{12}'] + \max[0,\ (1-\eta h)\widetilde{u}_3\widetilde{p}_{13}'] \tag{33}$$

• Similarly, the μ-functions for the second pair of lotteries, eq.(17), p.8, are:

$$\mu_2(h) = \max[0,\ (1-\eta h)\widetilde{u}_2\widetilde{p}_{22}] + \max[0,\ (1-\eta h)\widetilde{u}_3\widetilde{p}_{23}] \tag{34}$$

$$\mu_2'(h) = \max[0,\ (1-h)\widetilde{u}_2\widetilde{p}_{22}'] + \max[0,\ (1-h)\widetilde{u}_3\widetilde{p}_{23}'] \tag{35}$$

¶ Assume:

$$\widetilde{u}^T\widetilde{p}_1' < \widetilde{u}^T\widetilde{p}_1 \tag{36}$$

which implies that the expected-utility-maximizer would prefer L_1 over L_1'.

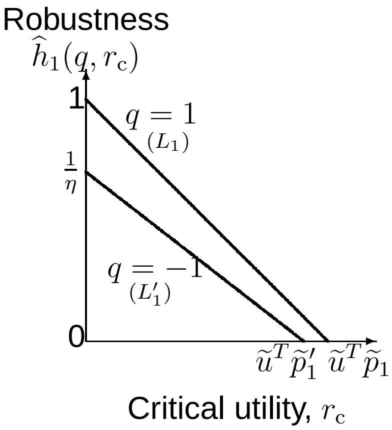

Figure 4: Robustness curves for the first pair of lotteries, eq.(16).

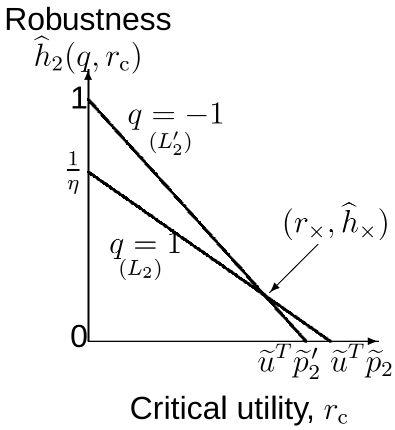

Figure 5: Robustness curves for the second pair of lotteries, eq.(17).

¶ 1st pair of lotteries, eq.(16):

- Robustness curves, based on eqs.(32) and (33), in fig. 4.
- Robust-satisficer and EU-maximizer agree: $L_1 \succ L_1'$, for different reasons.
- EU-maximizer: because of eq.(36) on p.11.
- Robust-satisficer: $\eta > 1$ so L_1 more robust to uncertainty in the perceived probabilities than L_1'.

¶ 2nd pair of lotteries, eq.(17):

- Robustness curves, based on eqs.(34) and (35), in fig. 5.
- Eqs.(21) and (36) imply:

$$\widetilde{u}^T\widetilde{p}_2' < \widetilde{u}^T\widetilde{p}_2 \tag{37}$$

- EU-maximizer: $L_2 \succ L_2'$.
- Robust-satisficer would not necessarily agree.
- The robustness curves cross at utility-aspiration $r_\times$ and at robustness $\widehat{h}_\times$.
- $r_\times$ is large and $\widehat{h}_\times$ is small:
  - since anticipated expected utility, $\widetilde{u}^T\widetilde{p}_2$ and $\widetilde{u}^T\widetilde{p}_2'$, are nearly the same.
  - and if η is substantially greater than unity.
- Thus L_2' is more robust than L_2 for $r_c < r_\times$ and $\widehat{h} > \widehat{h}_\times$.
- The robust-satisficing agent will choose lottery L_2' unless:
  - Very little robustness is needed or,
  - Very great utility is aspired to.

¶ Summary:

- Robust-satisficer with probability-uncertainty displays preference-reversal like most of Allais' subjects.
- EU-maximizer with estimated probabilities inconsistent with most of Allais' subjects.

3.3 Utility Uncertainty

¶ Uncertainty.

- The DM is uncertain only about the utility vector.
- The subjective meaning of the probabilities is perfectly clear.
- Info-gap model of uncertainty in u:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u : \ (u - \widetilde{u})^T W (u - \widetilde{u}) \le h^2 \right\}, \quad h \ge 0 \tag{38}$$

W is a known, real, symmetric, positive definite matrix.

¶ Decision vector:

q, eq.(30), p.10, for choosing between lotteries L_n and L'_n, specified in eqs.(16) and (17), p.8.

¶ Robustness:

$$\widehat{h}_n(q, r_c) = \max \left\{ h : \ \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} u^T \left[q \left(\widetilde{p}_n - \widetilde{p}'_n \right) \right] \right) \ge r_c \right\} \tag{39}$$

$\widetilde{p}_n$ and $\widetilde{p}'_n$ are probability vectors of lotteries L_n and L'_n respectively.

¶ Evaluating the robustness.

- Minimum in eq.(39) is:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} u^T \left[q \left(\widetilde{p}_n - \widetilde{p}'_n \right) \right] = q \widetilde{u}^T \left(\widetilde{p}_n - \widetilde{p}'_n \right) - h \underbrace{\sqrt{\left(\widetilde{p}_n - \widetilde{p}'_n \right) W^{-1} \left(\widetilde{p}_n - \widetilde{p}'_n \right)}}_{\xi_n} \tag{40}$$

- Robustness in eq.(39) is the least upper bound of the set of h-values for which:

$$q \widetilde{u}^T \left(\widetilde{p}_n - \widetilde{p}'_n \right) - h \xi_n \ge r_c \tag{41}$$

- Define $\widetilde{\Delta} = \widetilde{u}^T \left(\widetilde{p}_n - \widetilde{p}'_n \right) =$, anticipated utility-premium of lottery L_n over L'_n.
- Robustness of choice q, with required utility-premium r_c, is:

$$\widehat{h}(q, r_c) = \begin{cases} \dfrac{q \widetilde{\Delta} - r_c}{\xi_n} & \text{if } r_c \le q \widetilde{\Delta} \\ 0 & \text{else} \end{cases} \tag{42}$$

¶ Choices:

- Robust-satisficer: choose q so that $q \widetilde{\Delta} > 0$. That is:

$$L_n \succeq L'_n \quad \text{if and only if} \quad \widetilde{u}^T \widetilde{p}_n \ge \widetilde{u}^T \widetilde{p}'_n \tag{43}$$

- From eq.(22), p.9:

robust-satisficer will agree with expected-utility-maximizer in both lottery selections.

¶ Summary of sections 3.2 and 3.3.

• A robust-satisficing DM whose uncertainty is limited to the perceived probabilities will usually display the preference-reversal and axiom-violation displayed by some of Allais' subjects (section 3.2).

• A robust-satisficing DM with uncertainty only about the utilities of the prizes will behave just like an expected-utility-maximizer (section 3.3).

• In short, the robust-satisficing decision paradigm, combined with diversity of the agents' uncertainties, is consistent with Allais' famous experiments.

4 Info-Gap Analysis of Expected-Utility Risk Aversion

4.1 Expected-Utility Risk Aversion

¶ Consider prizes of value

$$w_1 > w_2 > w_3 \tag{44}$$

where

$$w_2 = P w_1 + (1 - P) w_3 \tag{45}$$

and $0 < P < 1$. These prizes can be won in either of the lotteries:

$$L: \quad \widetilde{p} \quad = \quad (P,\, 0,\, 1 - P)^T \tag{46}$$

$$L': \quad \widetilde{p}' \quad = \quad (0,\, 1,\, 0)^T \tag{47}$$

P and $1 - P$ are the probabilities of winning w_1 and w_3 respectively.
L is a gamble between high and low gain.
L' is a sure bet on an outcome which is the mean of the extremes.

¶ Utility function:

- $\widetilde{u}(w_i) =$ decision maker's personal utility from prize w_i.
- Define vector:

$$\widetilde{u} = (\widetilde{u}(w_1),\ \widetilde{u}(w_2),\ \widetilde{u}(w_3)) \tag{48}$$

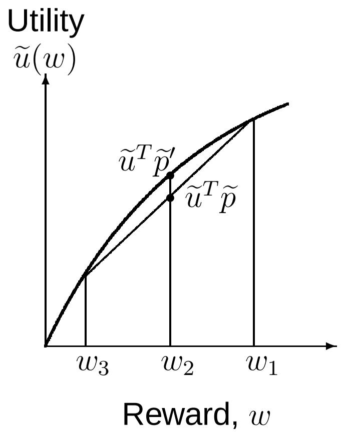

Figure 6: Concave estimated utility function. Risk aversion.

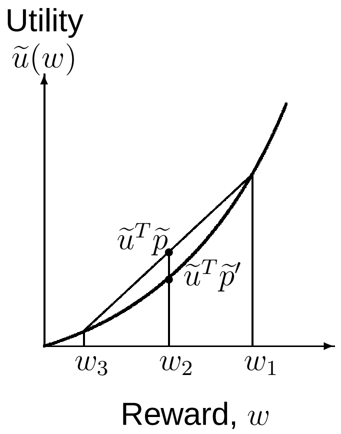

Figure 7: Convex estimated utility function. Risk proclivity.

¶ **Expected utility of L and L':**

$$\mathrm{E}(L) = P\widetilde{u}(w_1) + (1-P)\widetilde{u}(w_3) = \widetilde{u}^T\widetilde{p} \quad (49)$$

$$\mathrm{E}(L') = \widetilde{u}(w_2) \quad (50)$$

$$= \widetilde{u}[Pw_1 + (1-P)w_3] = \widetilde{u}^T\widetilde{p}' \quad (51)$$

¶ **Expected-utility preference:**

$$L \succ L' \quad \text{if and only if} \quad P\widetilde{u}(w_1) + (1-P)\widetilde{u}(w_3) > \widetilde{u}(w_2) \quad (52)$$

¶ **Risk aversion:**

- Prefer certainty-equivalent sure thing, w_2, over gamble between w_1 and w_3.
- Prefer L' over L.
- Fig. 6: Concave utility function.
- Degree of risk aversion: curvature of utility function.

¶ **Risk proclivity:**

- Prefer gamble between w_1 and w_3 over certainty-equivalent sure thing, w_2.
- Prefer L over L'.
- Fig. 7: Convex utility function.
- Degree of risk proclivity: curvature of utility function.

¶ **Expected-utility risk aversion:**

- Depends on knowing utilities and probabilities.
- Not usually suited for severe uncertainty.

¶ We will study risk sensitivity from an **info-gap perspective.**

4.2 Info-Gap Analysis with Uncertain Probabilities

¶ Uncertainty in subjective probabilities:

- $\widetilde{p}$ in L, eq.(46) : more complicated, less clear.
- $\widetilde{p}'$ in L', eq.(47): less complicated, more clear.
- Fractional-error info-gap models:
  - ◦ $\mathcal{L}(h, \widetilde{p})$ in eq.(28), p.10, for $\widetilde{p}$.
  - ◦ $\mathcal{L}'(h, \widetilde{p}')$ in eq.(29), p.10, for $\widetilde{p}'$.

$$\mathcal{L}(h, \widetilde{p}) = \left\{ p \in \mathcal{P}_3 : \left| \frac{p_i - \widetilde{p}_i}{\widetilde{p}_i} \right| \le \eta h, \ i = 1, \, 2 \right\}, \ h \ge 0 \tag{53}$$

$$\mathcal{L}'(h, \widetilde{p}') = \left\{ p \in \mathcal{P}_3 : \left| \frac{p_i - \widetilde{p}'_i}{\widetilde{p}'_i} \right| \le h, \ i = 1, \, 2 \right\}, \ h \ge 0 \tag{54}$$

$\eta > 1$ so $\mathcal{L}(h, \widetilde{p})$ is an expanded and shifted version of $\mathcal{L}'(h, \widetilde{p}')$.

¶ DM's goal:

- Robust-satisfice expected utility, $\widetilde{u}^T p$.
- $\widetilde{u}$ is known; probability distribution p is uncertain.
- Decision variable:

$$q = \begin{cases} 1 & \text{for choosing } L \\ -1 & \text{for choosing } L' \end{cases} \tag{55}$$

- Robustness of choice q between lotteries L and L', with critical utility r_c, p.10:

$$\widehat{h}(q, r_c) = \max \left\{ h : \left(\frac{1+q}{2} \min_{p \in \mathcal{L}(h, \widetilde{p})} \widetilde{u}^T p \ + \ \frac{1-q}{2} \min_{p \in \mathcal{L}'(h, \widetilde{p}')} \widetilde{u}^T p \right) \ge r_c \right\} \tag{56}$$

¶ Decisions:

- Suppose $\widetilde{u}^T\widetilde{p} > \widetilde{u}^T\widetilde{p}'$.
- Thus eq.(52) holds and utility-maximizing agent prefers:

$$\text{(risky) } L \succ L' \text{ (risk free)} \tag{57}$$

as in fig. 7 on p.17.

- However, robustness curves cross at utility-aspiration $r_\times$ and robustness $\widehat{h}_\times$, as in fig. 8.

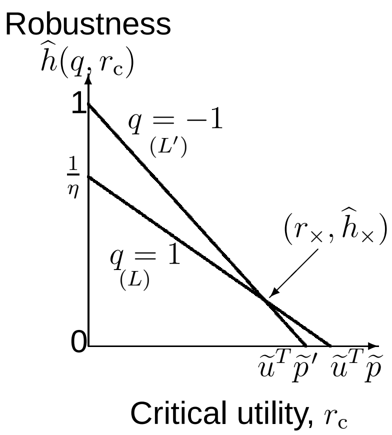

Figure 8: Robustness curves.

- If:

  $\widetilde{u}^T\widetilde{p}$ is not too different from $\widetilde{u}^T\widetilde{p}'$,

  or if

  η is substantially greater than one,

  then $r_\times$ is large and $\widehat{h}_\times$ is small,

  So robust-satisficer prefers:

$$\text{(risk-free) } L' \succ L \text{ (risky)} \tag{58}$$

in contrast to utility-maximizer with the same utility function.

¶ Risk aversion is expressed by:

- Demand for utility, as manifested in curvature of utility function.
- Demand for robustness, as manifested in robustness function.

5 Info-Gap Robust-Satisficing Foraging

¶ Is robustness a good (probabilistic) bet? What can we learn from animal behavior?

¶ Joint work with Dr. Yohay Carmel.

¶ Foraging decision:

- Animal currently feeding “here”.
- To move or not to move? That is the question.

§ Optimization model: **Maximize** energy intake.

§ Robust-satisficing model:

- **Satisfice** energy intake.
- **Maximize** robustness to uncertainty.

¶ **Variables:**

$G_0 =$ energy gain here (highly uncertain).

$G_1 =$ energy gain there (highly uncertain).

$\widetilde{G}_0$ = estimate of G_0.

$\widetilde{G}_1$ = estimate of G_1.

$C =$ cost of moving.

$T =$ remaining time.

¶ **System model:**

- Define decision variable:

$$q = \begin{cases} -1 & \text{moving} \\ 1 & \text{staying} \end{cases} \tag{59}$$

- Energy intake:

$$E(G, q) = \frac{1+q}{2} G_0 T + \frac{1-q}{2} (G_1 T - C) \tag{60}$$

¶ **Performance requirement:**

$$E(G, q) \geq E_{\rm c} \tag{61}$$

¶ Fractional error info-gap model:

$$\mathcal{U}(h) = \left\{ G_0, G_1 : \left| \frac{G_i - \widetilde{G}_i}{\widetilde{G}_i} \right| \le h, \ i = 0, 1 \right\}, \quad h \ge 0 \tag{62}$$

¶ Optimal foraging:

- Maximize best estimate of energy.
- Leave patch when:

$$\widetilde{G}_0 T < \widetilde{G}_1 T - C \tag{63}$$

¶ Robust satisficing:

- $E_{\min}$ = least acceptable energy.
- Reliably achieve $E_{\min}$.

¶ Robustness definition and derivation:

$$\widehat{h}(E_c, q) = \max \left\{ h : \left(\min_{G \in \mathcal{U}(h)} E(G, q) \right) \ge E_c \right\} \tag{64}$$

- $m(h)$ denotes inner minimum, occurring at minimal energy gains:

$$m(h) = \frac{1+q}{2}(1-h)\widetilde{G}_0 T + \frac{1-q}{2}[(1-h)\widetilde{G}_1 T - C] \tag{65}$$

$$= E(\widetilde{G}, q) - \left(\frac{1+q}{2}\widetilde{G}_0 T + \frac{1-q}{2}\widetilde{G}_1 T \right) h \tag{66}$$

- Equate $m(h)$ to E_c and solve for h to obtain robustness:

$$\widehat{h}(E_c, q) = \frac{E(\widetilde{G}) - E_c}{\frac{1+q}{2}\widetilde{G}_0 T + \frac{1-q}{2}\widetilde{G}_1 T} \tag{67}$$

or zero if this is negative. Specifically:

$$\widehat{h}(E_c, q = 1) = 1 - \frac{E_c}{\widetilde{G}_0 T} \tag{68}$$

$$\widehat{h}(E_c, q = -1) = 1 - \frac{E_c + C}{\widetilde{G}_1 T} \tag{69}$$

§ Two cases:

- Optimal to stay: $\widetilde{G}_1 T - C < \widetilde{G}_0 T$.
- Optimal to move: $\widetilde{G}_1 T - C > \widetilde{G}_0 T$.

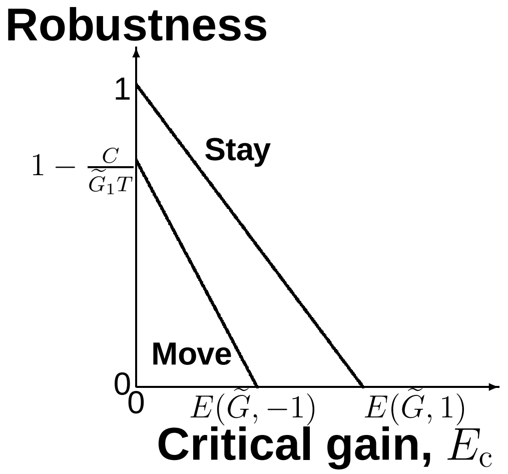

Figure 9: $\widetilde{G}_1T - C < \widetilde{G}_0T$.

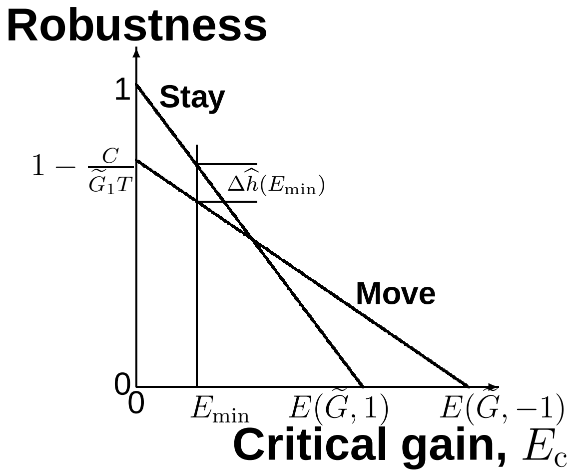

Figure 10: $\widetilde{G}_1T - C > \widetilde{G}_0T$.

§ Fig. 9:

- Nominally optimal to **stay**: $\widetilde{G}_1T - C < \widetilde{G}_0T$.
- Robust satisficer **also stays.**

§ Fig. 10:

- Nominally optimal to **move**: $\widetilde{G}_1T - C > \widetilde{G}_0T$.
- Robust satisficer **may also move**.

¶ **Preliminary lit. survey** of 34 studies

suggests support for robust-satisficing in many taxa.

¶ **Why optimize?**

More is better than less.

Hence most is best.

¶ **Why not optimize?**

(Even though most is best.)

¶ **Bounded rationality** (Simon):

- Poor information.
- Poor information processing.
- Optimization unreliable, infeasible.

¶ **Robust-satisficing** is proxy for **Probability of success:**

max Robustness $\Longrightarrow$ max Prob of Success

$\Longrightarrow$ max Prob of Survival

¶ **Darwin and satisficing:**

- Evolution:
  - ○ Survival of more fit over less fit (*O of S*).
  - ○ Don't optimize. Satisfice:
    - — Beat competition.
    - — Robustify against surprises.

¶ **Darwin and satisficing:**

- Species distribution:
  - ○ "Great fact" (*O of S*):

  Similar habitats in old & new worlds have
  "widely different . . . living productions!".
  - ○ Optimization:

  Similar phenotypes under similar constraints.
  - ○ Robust satisficing:

  Diversity from performance-sub-optimality.

¶ **Foraging animals seem to robust-satisfice.**

¶ **Questions:**

- Is this mental frailty of animals or evolutionary guile?
- It seems that satisficing is strategically advantageous.

6 Probability of Success

¶ Main idea:

- Robust-satisficing is sometimes a better bet than direct optimization.
- In this section we will discuss a theorem which indicates why this is true.

¶ Animals seem to robust-satisfice rather than to optimize. Thus robust-satisficing seems to be an evolutionarily successful strategy. We consider several examples.

- Invasive species:
  - ◦ Darwin:
    - — Invasive species introduced to a new region can dominate successful endemic species due to the newcomers' superior fitness.
    - — The successful endemics did not optimize. They just beat the local competition.
  - ◦ Simon: evolutionary success is by comparative advantage not universal optimality. 'Comparative advantage' is a satisficing strategy (don't optimize, just beat the competition),
  - ◦ Darwinian evolution is the survival of the more fit over the less fit, not necessarily of the *most* fit.
- Old and new world habitats:
  - ◦ Darwin: similar habitats in the Old and New Worlds have "widely different ... living productions!".
  - ◦ Optimization would tend to produce similar phenotypes under similar constraints.
  - ◦ Robust satisficing produces diversity due to the added degrees of freedom associated with performance-sub-optimality.
- Foraging by animals from numerous diverse taxa:

  sub-optimal but robust strategies preferred over performance-optimal decisions.
- Economic puzzles,
  - ◦ Home-bias paradox,
  - ◦ Equity premium puzzle,

  suggest that economic agents robust-satisfice rather than optimize.

¶ Engineering design: satisfy specifications; don't optimize performance.

6.1 A Simple Choice

¶ The choice:

- Choose 1 from among I options or alternatives.
- $v_i =$ value from option i.
- Large value better than small value.
- At least value $v_{\rm c}$ needed for survival.

¶ Uncertainty:

$\widetilde{v}_i =$ best-estimate of value of option i

$\sigma_i =$ estimated error of $\widetilde{v}_i$.

- Info-gap model:

$$\mathcal{U}(h, \widetilde{v}_i) = \left\{ v_i : \ |v_i - \widetilde{v}_i| \le h\sigma_i \right\}, \quad h \ge 0 \tag{70}$$

¶ Question: What strategy?

- Direct optimization. Maximize best-estimate of value:

$$i^\star = \arg\max_i \widetilde{v}_i \tag{71}$$

Why is direct optimization attractive:

- ∘ $i^\star$ has the best value if the estimates are correct.
- ∘ Even if the estimates err, hopefully option $i^\star$ is still better than others.

- Robust-satisficing:
  - ∘ Satisfice value.
  - ∘ Maximize robustness to uncertainty in value.
- Robustness:

$$\widehat{h}(i, v_{\rm c}) = \max \left\{ h : \ \left(\min_{v_i \in \mathcal{U}(h, \widetilde{v}_i)} v_i \right) \ge v_{\rm c} \right\} \tag{72}$$

- Robust-satisficing option:

$$\widehat{i}(v_{\rm c}) = \arg\max_i \widehat{h}(i, v_{\rm c}) \tag{73}$$

Why is robust-satisficing attractive: Maximize reliability of survival.

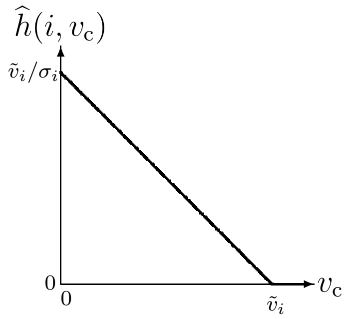

Figure 11: Robustness curve, eq.(74).

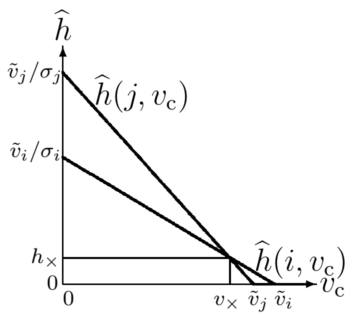

Figure 12: Comparison of two options: reversal of preferences.

¶ Robustness function, fig. 11:

$$\widehat{h}(i, v_c) = \begin{cases} \dfrac{\widetilde{v}_i - v_c}{\sigma_i} & \text{if } \widetilde{v}_i \geq v_c \\ 0 & \text{else} \end{cases} \tag{74}$$

- Trade-off of $\widehat{h}$ vs. v_c.
- No robustness of nominal best estimate:

$$\widehat{h}(i, v_c) = 0 \text{ if } \quad v_c = \widetilde{v}_i \tag{75}$$

¶ Two alternative options with parameters:

$$\widetilde{v}_i > \widetilde{v}_j \quad \text{and} \quad \frac{\widetilde{v}_i}{\sigma_i} < \frac{\widetilde{v}_j}{\sigma_j} \tag{76}$$

- Robustness curves in fig. 12.
- Eq.(76) is basic dilemma of DM. Option i has:
  - ○ higher estimated value than option j,
  - ○ larger estimation-uncertainty than j.
- Eq.(76) are necessary and sufficient for the robustness curves to cross in the positive quadrant, as in fig. 12.

¶ Compare robust-satisficing and direct optimizing choice between any options i and j.

- Based on fig. 12.
- Robust-satisficing and direct-optimizing **differ** if and only if:

$$\widetilde{v}_i > \widetilde{v}_j \quad \text{and} \quad \frac{\widetilde{v}_i}{\sigma_i} < \frac{\widetilde{v}_j}{\sigma_j} \quad \text{and} \quad v_c < v_\times \tag{77}$$

- They **agree** otherwise.

6.2 Probability of Survival: Standardization

¶ Which strategy,

- Robust-satisficing, eq.(73), p.25, or,
- Direct optimizing, eq.(71), p.25,

will have greater probability of survival?

¶ Probability of survival.

- Option i succeeds if its value is no less than the critical value:

$$v_i \geq v_{\mathrm{c}} \tag{78}$$

- $F_i(\cdot)$ denotes the cumulative probability distribution function of v_i.
- Probability of success for option i is:

$$P_{\mathrm{s}}(i) \;=\; \text{Prob}(v_i \geq v_{\mathrm{c}}) = 1 - F_i(v_{\mathrm{c}}) \tag{79}$$

¶ Standardization class of probability distributions.

- $f(x|q)$ is a pdf of a random variable x, where q is a vector of parameters of the pdf.
- $f(x|q)$ is a class of pdfs parametrized by q.
- Mean and variance of x are μ_x and σ_x^2. E.g. $q = (\mu_x, \sigma_x^2)$.
- Standardized random variable, with pdf $g(y)$, is:

$$y = (x - \mu_x)/\sigma_x \tag{80}$$

- If $g(y)$ is independent of q then this is a standardization class. That is, if all the standardized random variables in the class have the same pdf, then this is a standardization class.
- Standardization classes are quite common:
  - ◦ the normal, uniform, and exponential distributions all being examples.
  - ◦ The standardized distribution $g(y)$ may belong to the standardization class, e.g. normal and uniform, but this is not necessarily true, e.g. the exponential.

¶ Example: exponential distribution:

$$f(x|q) = q\mathrm{e}^{-qx}, \quad x \geq 0 \tag{81}$$

Moments:

$$\mathrm{E}(x|q) = \sigma(x|q) = \frac{1}{q} \tag{82}$$

Standardized variable:

$$y = \frac{x - \mathrm{E}(x|q)}{\sigma(x|q)} = qx - 1 \tag{83}$$

Standardized density by probability balance:

$$x = \frac{y+1}{q}, \quad \mathrm{d}x = \frac{1}{q}\mathrm{d}y \quad \Longrightarrow \quad g(y)\mathrm{d}y = p(x)\mathrm{d}x = \mathrm{e}^{-qx}q\mathrm{d}x = \mathrm{e}^{-(y+1)}\mathrm{d}y,\ y \geq -1 \tag{84}$$

Standardized density and cumulative distribution:

$$g(y) = \mathrm{e}^{-(y+1)},\ y \geq -1, \qquad G(y) = \int_{-1}^{y} g(z)\,\mathrm{d}z = 1 - \mathrm{e}^{-(y+1)} \tag{85}$$

$g(y)$ is a shifted exponential distribution.

¶ Probability of survival.

- Suppose v_i and v_j both belong to the same standardization class.
- $G(y) =$ cumulative probability distribution function of the standardized random variables.
- Probability of success for option i is:

$$P_{\mathrm{s}}(i) = \text{Prob}(v_i \geq v_{\mathrm{c}}) = \text{Prob}\left(\frac{v_i - \widetilde{v}_i}{\sigma_i} \geq \frac{v_{\mathrm{c}} - \widetilde{v}_i}{\sigma_i}\right) \tag{86}$$

$$= 1 - G\left(\frac{v_{\mathrm{c}} - \widetilde{v}_i}{\sigma_i}\right) \tag{87}$$

$$= 1 - G\left[-\widehat{h}(i, v_{\mathrm{c}})\right] \tag{88}$$

where eq.(88) results from eqs.(87) and (74) if $v_{\mathrm{c}} \leq \widetilde{v}_i$.

- We see that:

$$P_{\mathrm{s}}(i) > P_{\mathrm{s}}(j) \quad \text{if and only if} \quad \widehat{h}(i, v_{\mathrm{c}}) > \widehat{h}(j, v_{\mathrm{c}}) \tag{89}$$

- Theorem:
  - If the values of the options all belong to the same standardization class,
  - Then robust-satisficing is a better bet than direct-optimizating if eq.(77) hold.
  - The strategies coincide otherwise and then have the same probability of success.

¶ Note: We don't need to know the identity of the standardization class in order to exploit the proxy property.

6.3 Proxy Theorem

6.3.1 Formulation

¶ Preference relations.

- Variables: $r =$ decision, $q =$ uncertainty, $L(r, q) =$ loss function.
- Robustness function:

$$\widehat{h}(r, L_c) = \max \left\{ h : \left(\max_{q \in \mathcal{Q}(h, \widetilde{q})} L(r, q) \right) \leq L_c \right\} \tag{90}$$

- Robust-satisficing preferences:

$$r \succeq_r r' \quad \text{if} \quad \widehat{h}(r, L_c) > \widehat{h}(r', L_c) \tag{91}$$

- Probability of survival:

$$P_s(r, L_c) = \text{Prob}[L(r, q) \leq L_c] = \int_{L(r,q) \leq L_c} p(q) \, dq \tag{92}$$

$p(q)$ is unknown.

- Probabilistic preferences:

$$r \succ_p r' \quad \text{if} \quad P_s(r, L_c) > P_s(r', L_c) \tag{93}$$

¶ Critical question: when do the preferences in eqs.(91) and (93) agree?

¶ Why is this important?

If a proxy theorem holds, then robust-satisficers will survive with greater likelihood than direct optimizers.

¶ **General uncertain variable, u.**

- u is a general uncertain variable or function, with info-gap model $\mathcal{U}(h,\tilde{u})$.

Definition 1 *A scalar loss function $L(r,u)$ is* **reducible and separable** *if there is a scalar function $q(u)$, independent of r, such that $L(r,u)$ can be expressed as $L[r,q(u)]$.*

- $L(r,u)$ is reducible and separable if:
  $L(r,u)=\rho(r)q(u)$ or $L(r,u)=\rho(r)+q(u)$ or $\rho(r)^{q(u)}$ or $L(r,u)=q(u)^{\rho(r)}$,
  where $\rho(r)$ is a scalar independent of u and $q(u)$ is a scalar independent of r.
- $L(r,u)=\rho(r)q_1(u)+q_2(u)$ is separable but not reducible if q_1 and q_2 are distinct functions.
- $\mathcal{U}(h,\tilde{u})$ induces an info-gap model for a function $q(u)$:

$$\mathcal{Q}[h,q(\tilde{u})]=\{q:\ q=q(u),\ u\in\mathcal{U}(h,\tilde{u})\}\,,\quad h\geq 0 \tag{94}$$

If $\mathcal{U}(h,\tilde{u})$ and $q(u)$ do not depend on the decision, r, then neither does $\mathcal{Q}[h,q(\tilde{u})]$.

¶ **Scalar uncertain variable, q.**

- q is a scalar uncertain variable or function, with values in D and with info-gap model $\mathcal{Q}(h,\tilde{q})$. Define:

$$q^\star(h) = \max_{q\in\mathcal{Q}(h,\tilde{q})} q \tag{95}$$

$$q_\star(h) = \min_{q\in\mathcal{Q}(h,\tilde{q})} q \tag{96}$$

$$\mu(h) = \max_{q\in\mathcal{Q}(h,\tilde{q})} L(r,q) \tag{97}$$

Definition 2 *An info-gap model, $\mathcal{Q}(h,\tilde{q})$,* **expands upward continuously** *if, for any $\varepsilon>0$, there is a $\delta>0$ such that:*

$$|q^\star(h')-q^\star(h)|<\varepsilon \quad \textit{if} \quad |h'-h|<\delta \tag{98}$$

Continuous downward expansion *is defined similarly with $q_\star(\cdot)$ instead of $q^\star(\cdot)$.*

- Define the extremal values of the loss function $L(r,q)$ for scalar q's on D:

$$L_0(r)=\min_{q\in D} L(r,q),\quad L_1(r)=\max_{q\in D} L(r,q) \tag{99}$$

¶ Proxy Theorem.

Theorem 1 *Info-gap robustness is a proxy for probability of survival if the loss function is reducible and separable.*

Given:

• u *is a general info-gap variable with info-gap model* $\mathcal{U}(h, \widetilde{u})$*.*

• $L(r, u)$ *is a real-valued and scalar loss function which is reducible and separable and can be expressed* $L[r, q(u)]$ *where* $q(u)$ *is a single-valued scalar variable with connected range* D*.*

• *At any decision* r*, the loss function,* $L(r, q)$*, has extreme values* $L_0(r)$ *and* $L_1(r)$*, defined in eq.(99), either or both of which may be infinite.*

• $\mathcal{Q}(h, \widetilde{q})$ *is the info-gap model defined in eq.(94), whose range contains the domain* D*.*

• *The info-gap model* $\mathcal{Q}(h, \widetilde{q})$ *does not depend on the decision,* r*.*

• $\mathcal{Q}(h, \widetilde{q})$ *is a continuously upward expanding, or continuously downward expanding, info-gap model (or both).*

Then, *for any two decisions,* r_1 *and* r_2*, which each have positive robustness* $\widehat{h}(r_i, L_\mathrm{c})$ *to uncertainty in* u*, for* $L_\mathrm{c} \in [L_0(r_i),\ L_1(r_i)]$*,* $i = 1, 2$*:*

$$\widehat{h}(r_1, L_\mathrm{c}) \ge \widehat{h}(r_2, L_\mathrm{c}) \quad \textit{implies} \quad P_\mathrm{s}(r_1, L_\mathrm{c}) \ge P_\mathrm{s}(r_2, L_\mathrm{c}) \tag{100}$$

and

$$P_\mathrm{s}(r_1, L_\mathrm{c}) > P_\mathrm{s}(r_2, L_\mathrm{c}) \quad \textit{implies} \quad \widehat{h}(r_1, L_\mathrm{c}) > \widehat{h}(r_2, L_\mathrm{c}) \tag{101}$$

¶ Proxy Theorem: Corollaries.

• Define:

∘ $R =$ the set of all feasible decisions.

∘ $\widehat{R}(L_\mathrm{c}) =$ the set of feasible decisions which maximize the robustness and satisfice the loss at L_c.

∘ $R^\star(L_\mathrm{c}) =$ the set of feasible decisions which maximize the probability of survival while satisficing the loss at L_c.

∘ Explicitly:

$$\widehat{R}(L_\mathrm{c}) = \left\{ r \in R:\ r = \arg\max_{r \in R} \widehat{h}(r, L_\mathrm{c}) \right\} \tag{102}$$

$$R^\star(L_\mathrm{c}) = \left\{ r \in R:\ r = \arg\max_{r \in R} P_\mathrm{s}(r, L_\mathrm{c}) \right\} \tag{103}$$

Corollary 1 *Any decision which maximizes the robustness also maximizes the probability of survival.*

Given *the conditions of theorem 1 and that neither* $\widehat{R}(L_\mathrm{c})$ *nor* $R^\star(L_\mathrm{c})$ *is empty,* **then:**

$$\widehat{R}(L_\mathrm{c}) \subseteq R^\star(L_\mathrm{c}) \tag{104}$$

The following corollary is an important special case of theorem 1.

Corollary 2 *Info-gap robustness to a scalar variable is a proxy for probability of survival for any loss function.*
Given *the conditions of theorem 1 and that u is a scalar variable and that $L(r, q)$ is an* arbitrary *scalar loss function,* **then** *relations (100) and (101) hold.*

6.3.2 Simple Examples

We now consider a series of simple examples of the info-gap robustness function as a proxy for the probability of survival. The first five are based on corollary 2, and the last two depend on the full theorem 1.

Example 1 *Uncertain Bayesian mixing of two models.* An agent must make a choice, r, where the outcome depends on which of two models of the world, A and B, is true. For instance, a central-bank economist might wish to set a vector of interest rates, r, so that the expected inflation, $L(r, q)$, is less than a critical value, $L_{\rm c}$, where q is the probability that model A is true. The agent believes that the economy is described either by a neo-Keynesian model (A) or a monetarist model (B). The agent's degree of belief that model A is true is $\widetilde{q}$, and that model B is true is $1 - \widetilde{q}$. However, $\widetilde{q}$ is highly uncertain, and its uncertainty is represented by an info-gap model $\mathcal{Q}(h, \widetilde{q})$. If the conditions of corollary 2 hold, in particular, if $\mathcal{Q}(h, \widetilde{q})$ expands continuously, then the robustness of any choice, r, is a proxy for the probability of success of that choice. That is, any change in the choice which augments the robustness cannot reduce the probability that the requirement, $L(r, q) \le L_{\rm c}$, is met. We will consider a variation on this example in section 6.3.3. ∎

Example 2 *Risky and risk-free assets.* An agent must determine the fraction r of a budget to invest in risky assets, the remainder to be invested in risk-free bonds. The uncertainty in the return to the risky asset, q, is represented by an info-gap model $\mathcal{Q}(h, \widetilde{q})$. $L_{\rm c}$ is a 'reservation' return. That is, the funds will be diverted to an alternative use if the returns, $L(r, q)$, are not confidently anticipated to exceed $L_{\rm c}$. If the conditions of corollary 2 hold, then the robustness of risky fraction r is a proxy for the probability that the expected returns, $L(r, q)$, will exceed $L_{\rm c}$. Choosing r to maximize the robustness against uncertainty in q will also maximize the probability that $L(r, q)$ exceeds $L_{\rm c}$. ∎

Example 3 *Foraging in an uncertain environment: Is the grass really greener?* An agent must choose how long, r, to stay in current conditions before moving to a new but uncertainly known location. For instance, a foraging bug (or a human job seeker) must decide how long to continue eating on its current bush before moving to a new and unknown bush. The relative productivity of current and new bushes is $\widetilde{q}$, whose uncertainty is quantified by an info-gap model $\mathcal{Q}(h, \widetilde{q})$. The survival depends on total gain $L(r, q)$ (of energy in the case of the bug) being no less than a critical value $L_{\rm c}$ (needed for the bug to survive the night). If the conditions of corollary 2 hold, then the robustness of duration r is a proxy for the probability that the total gain will satisfy the survival requirement, $L(r, q) \ge L_{\rm c}$. ∎

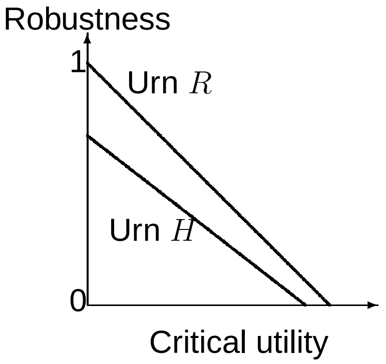

Figure 13: Robustness curves for the first experiment.

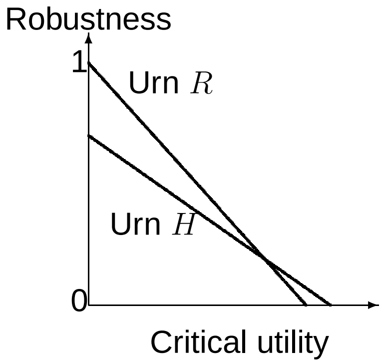

Figure 14: Robustness curves for the second experiment.

Example 4 *Ellsberg's 'paradox' and crossing robustness curves.* Mas-Colell *et al* (1995, p.207) explain the Ellsberg paradox (Ellsberg, 1961) in terms of perceptions of uncertainty which have a natural formulation with info-gap decision theory as developed in (Ben-Haim, 2006, section 11.1).

Ellsberg's observation, as adapted by Mas Colell *et al*, begins with two urns, R and H, where R contains a well shaken mixture of 49 white and 51 black balls, while H contains an unknown mixture of 100 white and black balls. Two balls are chosen randomly, one from each urn, and their colors are not revealed. The agent must choose one of these balls in each of two experiments. In the first experiment the agent wins \$1000 only if the selected ball is black. Most participants choose the ball from R, suggesting that their subjective probability for a white ball from H is greater than 0.49. In the second experiment the agent wins \$1000 only if the selected ball is white. Again most respondents choose the ball from R even though they are aware that the probability of a white ball from R is precisely 0.49. Ellsberg's 'paradox' is the agent's anomalous disregard for the fact that the subjective probability for white balls is presumably greater for urn H than for urn R.

The info-gap explanation of this behavior supposes that the agent wishes to reliably satisfice the expected utility at no less than the reservation value L_{c}. The dominant uncertainty, q, is the unknown fraction of white balls in urn H, represented by an info-gap model $\mathcal{Q}(h, \tilde{q})$. If the conditions of corollary 2 hold, then the robustness of the choice between the urns is a proxy for the probability of satisficing the expected utility. An info-gap robustness analysis for this problem (Ben-Haim, 2006, section 11.1) yields the robustness curves of figs. 13 and 14.

In the first experiment (win on black) the robustness curve for choosing urn R lies strictly above the robustness curve for choosing urn H, as in fig. 13. R is preferred both in terms of expected return and robustness. In the second experiment (win on white) the robustness curves for the two urns cross at high expected utility (very near the utility-axis), as in fig. 14, indicating that R will be favored in most cases, as observed. Most of the agents in Ellsberg's experiments choose robustness-maximizing urns, which, according to our proxy theorem, is equivalent to maximizing the probability of satisficing the expected utility. There is nothing paradoxical in Ellsberg's observations: most people are robust-satisficers. ■

Example 5 *Multiple requirements.* An agent must make decisions encoded in a vector r where the outcome must satisfy multiple requirements: $L_i(r, q) \geq L_{c,i}$, $i = 1, \ldots, I$. Uncertainty of the parameter q is represented by an info-gap model, $\mathcal{Q}(h, \widetilde{q})$. The robustness for the ith requirement alone is $\widehat{h}_i(r, L_{c,i})$. If the conditions of corollary 2 hold, then each individual robustness function, $\widehat{h}_i(r, L_{c,i})$, proxies for the probability of satisfying the corresponding requirement. Any change in the decision r which enhances all the robustness functions also enhances the probabilities of satisfying each of the requirements. ■

Example 6 *Forecasting an uncertain dynamical system.* An agent chooses a mathematical model with parameters r to forecast the behavior of a dynamical system whose true dynamics are determined by uncertain parameters and functions u. The forecast error is $L(r, u) = |L_1(r) - L_2(u)|$ where each function L_i is scalar-valued and independent of the argument of the other function. This forecast error is a reducible and separable loss function. The best estimate of the true dynamical parameters is $\widetilde{u}$ whose uncertainty is represented by an info-gap model $\mathcal{U}(h, \widetilde{u})$. If the conditions of theorem 1 hold, then the robustness of forecasting with model r proxies for the probability that the forecast error is no greater than L_c. Note that the forecast model, parameterized with r, may be fundamentally different from (e.g. much simpler than) the true model. Nonetheless, any change in r which enhances the robustness also increases the probability of forecast success. Furthermore, a model parameterized with r may be very different from a statistically estimated model. Nonetheless, if the r's robustness exceeds the robustness of the statistically estimated model (due to crossing of their robustness curves) then model r has higher probability of successful forecasting. We will consider an example of forecasting in greater depth in section 6.3.4. ■

Example 7 *Managing exogenous uncertainties.* Consider a system which an agent wishes to control by taking action r. A vector u of uncertain quantities is *exogenous* to the system if u's info-gap model, $\mathcal{U}(h, \widetilde{u})$, does not depend on action r. Suppose the scalar loss function, $L(r, u)$, is reducible and separable, for instance if $L(r, u) = L_1(r) + L_2(u)$ or if $L(r, u) = L_1(r)L_2(u)$. If the other conditions of theorem 1 hold as well, then the info-gap robustness for satisficing the loss at a value L_c is a proxy for the probability that the loss will not exceed L_c. ■

6.3.3 Example: Model Mixing

In this section we will explore a variation of example 1 discussed in section 6.3.2.
Consider a system for which the loss resulting from decision r is the following scalar function:

$$L(r,q) = A(r) + qB(r) \tag{105}$$

where $A(r)$ and $B(r)$ are known functions but q is an uncertain parameter. For instance, $A(r)$ may be a linear model and $B(r)$ a non-linear term of uncertain weight. Or, $A(r)$ and $B(r)$ may be two competing models of loss where q is the uncertain measure of their relative importance. There are no constraints on $A(r)$ and $B(r)$ other than that they are scalar functions and $B(r) \neq 0$. The decision r may be a scalar, vector, or function.
The best estimate of q is $\widetilde{q}$, which is non-zero,[1] and the info-gap model for uncertainty in this estimate is:

$$\mathcal{Q}(h, \widetilde{q}) = \left\{ q : \left| \frac{q - \widetilde{q}}{\widetilde{q}} \right| \leq h \right\}, \quad h \geq 0 \tag{106}$$

The robustness of decision r, with the requirement that the loss not exceed $L_{\rm c}$, is:

$$\widehat{h}(r, L_{\rm c}) = \max \left\{ h : \left(\max_{q \in \mathcal{Q}(h, \widetilde{q})} L(r,q) \right) \leq L_{\rm c} \right\} \tag{107}$$

An explicit expression in the present case is:

$$\widehat{h}(r, L_{\rm c}) = \begin{cases} 0 & \text{if } L_{\rm c} < L(r, \widetilde{q}) \\ \dfrac{L_{\rm c} - L(r, \widetilde{q})}{\widetilde{q}|B(r)|} & \text{else} \end{cases} \tag{108}$$

Consider two decisions, r_1 and r_2, for which:

$$L(r_1, \widetilde{q}) < L(r_2, \widetilde{q}) \quad \text{and} \quad |B(r_1)| > |B(r_2)| \tag{109}$$

That is, r_2 has greater estimated loss (left relation) but its robustness curve is steeper (right relation). This implies that the robustness curves for these decisions will cross. See fig. 15.

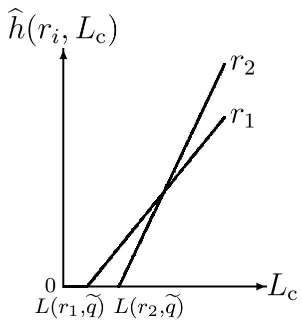

Figure 15: Crossing robustness curves.

[1]This example is readily modified for the case that $\widetilde{q} = 0$.

The conditions of corollary 2 hold, so robustness is a proxy for the probability that the loss, $L(r, q)$, will not exceed $L_{\rm c}$. While we know nothing about the probability distribution of q, we are able to evaluate the robustness, $\widehat{h}(r, L_{\rm c})$, to uncertainty in q. If the conditions in eq.(109) do not hold then one robustness curve lies strictly above the other, implying that robustness- and probability-preferences, eqs.(91) and (93), agree at all values of critical loss $L_{\rm c}$. If eq.(109) does hold for a given pair of decisions, then we are able to determine the value of critical loss, $L_{\rm c}$, at which their robustness curves cross, implying reversal of preference between these decisions based on value of the critical loss. We are not able to evaluate the probability of satisficing the loss, but we can always determine which, among any collection of decisions, has the greatest probability of satisficing the loss at any given level $L_{\rm c}$.

6.3.4 Example: Forecasting

In this section we will explore a specific realization of example 6 discussed in section 6.3.2. Forecasting, in a broad sense, is done by meteorologists, stock brokers, foraging animals chasing evasive prey, job-hunters, vote-seekers, and so on. In many situations historical data is limited or incorrect, contextual information may be available but not manifested in historical data, future “rules of the game” may deviate systematically from the past, and probabilistic information is lacking. The probability that the forecast is accurate to within a given margin is critical, though the probability distribution of the uncertain quantities is not known. If an info-gap model represents the uncertainty, then the robustness of a forecasting algorithm is a proxy for the probability that the forecast is accurate to a specified extent. This means that the probability of success can be maximized by maximizing the info-gap robustness. We illustrate that here.

Consider a linear regression of a scalar variable x against a vector y of N measurements, with regression (or model) coefficients u:

$$x = u^T y \tag{110}$$

where the superscript T implies matrix transposition. We will formulate a model for forecasting x, based on measured y, when u is uncertain.

The best estimate of u is $\widetilde{u}$, based on historical data. However, the future evolution of the system may deviate systematically from the historical behavior. We consider the following info-gap model for uncertainty in the elements of u:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u : \ \widetilde{u}_i - h v_i \le u_i \le \widetilde{u}_i + h w_i, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{111}$$

where the v_i and w_i are non-negative and known. In particular, we are interested in *asymmetric uncertainty:* $v_i \neq w_i$ for at least some of the i’s, implying that the uncertainty-intervals for the corresponding coefficients u_i grow asymmetrically with increasing horizon of uncertainty h.

Asymmetric uncertainties based on contextual information are common under severe uncertainty. Dramatic political events may indicate that market variables y_i will tend to impact outcomes (x) more in one direction than another, though actual data are not yet available. Industrial growth suggests that global warming will increase, but there is great dispute about the extent of increase. The sudden appearance of a single insect on a bush may signal to individuals of other species, currently feeding on that bush, that strong competition may shortly emerge though the insect horde is not yet discernable. All these situations have asymmetric uncertainty which can be encoded in an info-gap model such as eq.(111).

Consider a simplified forecasting model, based on M of the N variables in y:

$$x_{\rm f} = r^T S y \tag{112}$$

where S is an $M \times N$ "selector" matrix whose rows are those M rows of the $N \times N$ identity matrix which correspond to those elements of y upon which the forecast is based. The choice of a low-dimensional model, such as eq.(112) if $M < N$, may be motivated by the limited computational ability of the agent or by the agent's severe uncertainty about the contribution of some of the elements of y. The agent's problem is to choose which elements of y to include and how to weight them.

The loss function is the forecast error:

$$L(r, u) = |r^T S y - u^T y| \tag{113}$$

This loss function is reducible and separable in the uncertain u with $q(u) = u^T y$. The forecaster must choose the coefficient-vector r and the selector-matrix S so that the forecast errs no more than $L_{\rm c}$. The conditions of theorem 1 hold, so the info-gap robustness is a proxy for the probability of forecasting with specified accuracy.

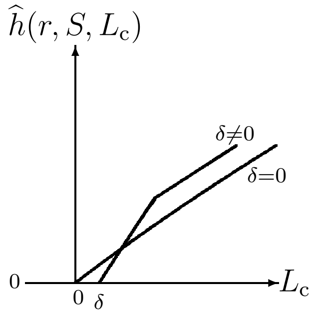

Figure 16: Robustness functions with $\delta = 0$ and $\delta \neq 0$.

The robustness function is illustrated in fig. 16. Define $\delta = x_{\rm f} - \widetilde{u}^T y$, which is the discrepancy between the forecasting model chosen by the agent, $x_{\rm f}$, and the best-estimate of the future behavior of the system, $\widetilde{u}^T y$. One would naively expect that $\delta = 0$ is the best-bet forecast. This is indeed the case if one aspires to very accurate forecast: requiring the forecast error, $L_{\rm c}$, to be very small. However, the robustness of achieving very small error is also very small because of the trade-off between robustness and forecast-error, expressed by the positive slope of the curves in fig. 16. If one can tolerate somewhat larger error then, as seen in fig. 16, the robustness is greater with $\delta \neq 0$ than with $\delta = 0$. The question is to choose δ: how sub-optimal should the forecasting model, $x_{\rm f}$, be? In light of theorem 1 we see that robustness is a proxy for probability of forecast success. Thus, for $L_{\rm c}$ greater than the value at which the curves cross in fig. 16, the probability of forecast success is better with $\delta \neq 0$ than with $\delta = 0$. Since the robustness curves can be evaluated without any probabilistic information, the agent can choose a forecasting model, $x_{\rm f} = r^T S y$, with maximum robustness, which maximizes the probability of forecasting with specified accuracy.

7 Is Bin Laden at Abbottabad?

¶ **Background material:** Jeffrey A. Friedman and Richard Zeckhauser, 2015, Handling and mishandling estimative probability: Likelihood, confidence, and the search for Bin Laden, *Intelligence and National Security,* Vol. 30, No. 1, 77–99.

¶ **The questions:**

• Several different intelligence experts have estimated the probability that Osama Bin Laden (OBL) is living at a specific location in Abbotabad, Pakistan (ABB). The estimates diverge widely: 0.4 to 0.8.

• President Obama must decide whether or not to attack that site.

• Part of his decision hinges on these estimates.

• How to process these subjective "estimative probabilities"?

• How to choose between the various options: attack, don't attack, gather more data?

7.1 Subjective Probability

¶ **Estimative probability for one analyst:**

• You are an intelligence expert. You have internalized lots of data and info.

• What is your subjective probability (estimative probability) that OBL is at ABB?

¶ **Estimative probability for one analyst: Ramsey lottery.**

• $p =$ your estimative probability that OBL is at ABB. We want to estimate p.

• First lottery, L_1:

 ◦ If OBL **is** at ABB, then you will win a **large prize**.

 ◦ If OBL **is not** at ABB, then you will win **nothing**.

 What odds do you put on OBL being at ABB?

• Second lottery, L_2:

 ◦ An urn contains 1000 thoroughly mixed balls: 300 red and 700 black.

 ◦ You will blindly draw a ball and:

  — win a **large prize** if the ball is black.

  — win **nothing** if the ball is red.

• **Question:** Which lottery do you prefer?

 ◦ If you prefer L_1 then $p > 0.7$. Repeat with 1000 balls, more than 700 black.

 ◦ If you prefer L_2 then $p < 0.7$. Repeat with 1000 balls, less than 700 black.

• Continue this until you are indifferent between L_1 and L_2 when $n =$ # black balls.

• Your estimative probability is $p = n/1000$.

¶ Combining estimative probabilities for 2 analysts with equal credibilities: Example.

- $p_1 = 0.4 =$ estimative probability of analyst 1 that OBL is at ABB.
- $p_2 = 0.8 =$ estimative probability of analyst 2 that OBL is at ABB.
- $c_i = 0.5 =$ relative credibility of analyst i: the probability that i's estimate is right, $i = 1,\ 2$.
- How to combine this information? Compound lottery:
- Two urns, each with 1000 thoroughly mixed red and black balls:
  - U_1 has 600 red and 400 black balls.
  - U_2 has 200 red and 800 black balls.
- You will blindly choose an urn (equal prob for each urn), and blindly choose a ball, and:
  - win a **large prize** if the ball is black.
  - win **nothing** if the ball is red.
- The probability of black (and large prize) is $p_{\mathrm{cmb}} = 0.5 \times 0.4 + 0.5 \times 0.8 = 0.6$.
- **Any other estimate would be inconsistent:**
  - $p_{\mathrm{cmb}} > 0.6$ would give analyst 2 extra credibility.
  - $p_{\mathrm{cmb}} < 0.6$ would give analyst 1 extra credibility.

¶ Combining estimative probabilities for N analysts with different credibilities.

- $p_i =$ estimative probability of analyst i that OBL is at ABB, $i = 1, \ldots, N$.
- $c_i =$ relative credibility of analyst i, where $c_i \geq 0$ and $\sum_{i=1}^{N} c_i = 1$.
- How to combine this information? Compound lottery:
- N urns, each with 1000 thoroughly mixed red and black balls:
  U_i has $1000(1 - p_i)$ red and $1000 p_i$ black balls, $i = 1, \ldots, N$.
- You will choose an urn with probability c_i for urn i, and blindly choose a ball, and:
  - win a **large prize** if the ball is black.
  - win **nothing** if the ball is red.
- The probability of black (and large prize), which is the combined estimative probability that OBL is at ABB is:

$$p_{\mathrm{cmb}} = \sum_{i=1}^{N} c_i p_i \tag{114}$$

- Any other p_{cmb} would be inconsistent with the estimates p_i and the credibilities c_i.

¶ How to assess confidence in the combined estimate?

- Friedman and Zeckhauser suggest **responsiveness**:

"have analysts not merely state estimative probabilities, but also to explain how much those assessments might change in light of further intelligence. This attribute might be termed an estimate's responsiveness." (p.92)

- This is closely related to—but different from—**info-gap robustness**:
  How much can the estimates change without altering the decision.
- Low responsiveness is similar to high robustness.
- Confidence is combined estimate is high if:
  - Responsiveness is low.
  - Robustness is high.

7.2 Info-Gap Robustness Analysis

¶ **System model:**

- $p =$ vector of estimative probabilities for N analysts.
- $c =$ vector of credibilities for N analysts.
- $\widetilde{p} =$ known specific realization of p for given analysts in specified conditions.
- $\widetilde{c} =$ known specific realization of c for given analysts in specified conditions.
- $p_{\text{cmb}}(p, c) = c^T p =$ combined probability that OBL is at ABB, from eq.(114).

¶ $\mathcal{U}(h) =$ Info-gap model of uncertainty for p and c.

¶ **Decision variable:**

$$A = \begin{cases} 0, & \text{Don't attack ABB} \\ 1, & \text{Attack ABB} \end{cases} \tag{115}$$

¶ **Binary decision with nominal estimates:**

- We will choose $A = 0$ if:

$$p_{\text{cmb}}(\widetilde{p}, \widetilde{c}) \le p_{\text{cr}} \tag{116}$$

- We will choose $A = 1$ if:

$$p_{\text{cmb}}(\widetilde{p}, \widetilde{c}) \ge p_{\text{cr}} \tag{117}$$

p_{cr} is a user-chosen critical probability.

§ **How confident are we in the decision?**

¶ Another possibility is to defer decision and gather more information.

¶ **Ternary decision with nominal estimates:**

- We will choose $A = 0$ if:

$$p_{\rm cmb}(\widetilde{p}, \widetilde{c}) \le p_{\rm cr,1} \tag{118}$$

- We will gather more information before deciding what to do if:

$$p_{\rm cr,1} \le p_{\rm cmb}(\widetilde{p}, \widetilde{c}) \le p_{\rm cr,2} \tag{119}$$

- We will choose $A = 1$ if:

$$p_{\rm cmb}(\widetilde{p}, \widetilde{c}) \ge p_{\rm cr,2} \tag{120}$$

¶ **We will consider the binary decision algorithm** of eqs.(116) and (117).

¶ **Robustness:** max horizon of uncertainty up to which decision does not change.

¶ **Robustness against missed opportunity** when $A = 0$:

$$\widehat{h}_0(p_{\rm cr}) = \max\left\{h:\ \left(\max_{p,c\in\mathcal{U}(h)} p_{\rm cmb}(p,c)\right) \le p_{\rm cr}\right\} \tag{121}$$

¶ **Robustness against erroneous attack** when $A = 1$:

$$\widehat{h}_1(p_{\rm cr}) = \max\left\{h:\ \left(\min_{p,c\in\mathcal{U}(h)} p_{\rm cmb}(p,c)\right) \ge p_{\rm cr}\right\} \tag{122}$$

¶ **Inverse robustness functions:**

- $m_0(h) =$ inner maximum in eq.(121). Inverse of $\widehat{h}_0(p_{\rm cr})$.
- $m_1(h) =$ inner minimum in eq.(122). Inverse of $\widehat{h}_1(p_{\rm cr})$.
- Zeroing:

$$m_0(0) = p_{\rm cmb}(\widetilde{p}, \widetilde{c}) = m_1(0) \tag{123}$$

- Trade offs:
  - ∘ $m_0(h)$ increases as h increases.
  - ∘ $m_1(h)$ decreases as h increases.

¶ **Useful function:**

$$x^+ = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } 1 < x \end{cases} \tag{124}$$

¶ **Robustness functions, example with 2 analysts:**

- The info-gap model for N analysts:

$$\mathcal{U}(h) = \left\{ p, c: \ 0 \le p_i \le 1, \ \left| \frac{p_i - \widetilde{p}_i}{\widetilde{p}_i} \right| \le h, \ c_i \ge 0, \sum_{i=1}^{N} c_i = 1, \left| \frac{c_i - \widetilde{c}_i}{\widetilde{c}_i} \right| \le h, \ \forall i \right\}, \quad h \ge 0 \tag{125}$$

- The combined estimative probability for 2 analysts is:

$$p_{\rm cmb}(p, c) = c^T p = c_1 p_1 + (1 - c_1) p_2 \tag{126}$$

- **Without loss of generality assume:** $\widetilde{p}_1 \ge \widetilde{p}_2$.
- Consider $m_0(h)$: **inverse robustness against missed opportunity.** It occurs for:

$$p_1 = [(1+h)\widetilde{p}_1]^+, \quad p_2 = [(1+h)\widetilde{p}_2]^+, \quad c_1 = [(1+h)\widetilde{c}_1]^+ \tag{127}$$

So:

$$m_0(h) = [(1+h)\widetilde{p}_1]^+ [(1+h)\widetilde{c}_1]^+ + \Big(1 - [(1+h)\widetilde{c}_1]^+\Big) [(1+h)\widetilde{p}_2]^+ \tag{128}$$

- **Special case:**

$$\widetilde{p}_1 \ge \widetilde{c}_1 \ge \widetilde{p}_2 \tag{129}$$

- Define:

$$h_{p_1} = \frac{1}{\widetilde{p}_1} - 1, \quad h_{p_2} = \frac{1}{\widetilde{p}_2} - 1, \quad h_{c_1} = \frac{1}{\widetilde{c}_1} - 1, \tag{130}$$

From eq.(129):

$$h_{p_1} \le h_{c_1} \le h_{p_2} \tag{131}$$

- From eq.(128) we now see that the inverse robustness is:

$$m_0(h) = \left\{ \begin{array}{cl} (1+h)^2 \widetilde{p}_1 \widetilde{c}_1 + (1+h)[1 - (1+h)\widetilde{c}_1]\widetilde{p}_2 & \text{if } h < h_{p_1} \\ (1+h)\widetilde{c}_1 + (1+h)[1 - (1+h)\widetilde{c}_1]\widetilde{p}_2 & \text{if } h_{p_1} \le h \le h_{c_1} \\ 1 & \text{if } h_{c_1} < h \end{array} \right. \tag{132}$$

This inverse robustness function is shown in fig. 17 illustrating zeroing and trade off.

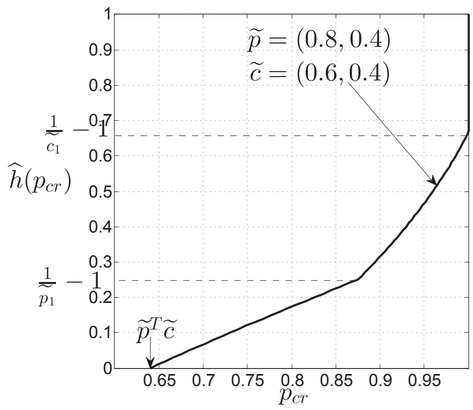

Figure 17: Robustness function from eq.(132).

¶ **Continue previous example, and consider 2 options:**

- Current state of knowledge is $(\widetilde{p}, \widetilde{c})$. Possible future state of knowledge is $(\widetilde{p}', \ \widetilde{c}')$:

$$\widetilde{p}, \ \widetilde{c} : \ \widetilde{p}_1 \geq \widetilde{c}_1 \geq \widetilde{p}_2 \quad \text{and} \quad \widetilde{p}', \ \widetilde{c}' : \widetilde{p}'_1 \geq \widetilde{c}'_1 \geq \widetilde{p}'_2 \tag{133}$$

where:

$$\widetilde{p}^T\widetilde{c} > (\widetilde{p}')^T\widetilde{c}' \quad \text{and} \quad \frac{1}{\widetilde{c}'_1} - 1 < \frac{1}{\widetilde{c}_1} - 1 \quad \Longleftrightarrow \widetilde{c}'_1 > \widetilde{c}_1 \tag{134}$$

- Left relation implies greater $p_{\rm cmb}$ for non-prime: nominal preference for non-prime option.
- Right relation implies greater confidence in prime option.
- The robustness curves for the two options **cross one another** as shown in fig. 18.
- This implies **preference reversal** between the options.

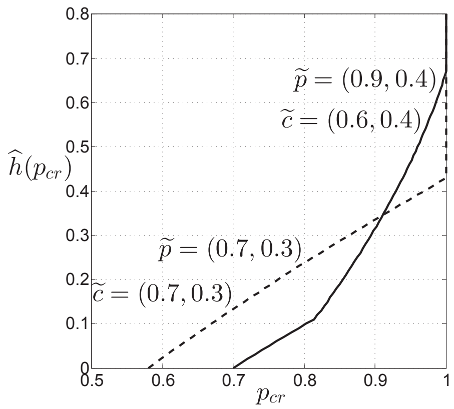

Figure 18: Robustness functions from eq.(132) for the two options in eqs.(133) and (134).

¶ **Continue previous example with one difference, and consider 2 options:**

- Current state of knowledge is $(\widetilde{p}, \widetilde{c})$. Possible future state of knowledge is $(\widetilde{p}', \ \widetilde{c}')$:
- **Retain** eq.(133):

$$\widetilde{p}, \ \widetilde{c} : \ \widetilde{p}_1 \geq \widetilde{c}_1 \geq \widetilde{p}_2 \quad \text{and} \quad \widetilde{p}', \ \widetilde{c}' : \widetilde{p}'_1 \geq \widetilde{c}'_1 \geq \widetilde{p}'_2 \tag{135}$$

Retain left side but **alter** right side of eq.(134):

$$\widetilde{p}^T\widetilde{c} > (\widetilde{p}')^T\widetilde{c}' \quad \text{and} \quad \frac{1}{\widetilde{c}_1'} - 1 > \frac{1}{\widetilde{c}_1} - 1 \quad \Longleftrightarrow \widetilde{c}_1' < \widetilde{c}_1 \tag{136}$$

- Left relation implies greater p_{cmb} for non-prime: nominal preference for non-prime option.
- Right relation implies greater confidence in non-prime option.
- The robustness curves for the two options **do not cross one another** as shown in fig. 19.
- This implies **robust dominance** of one of the options.

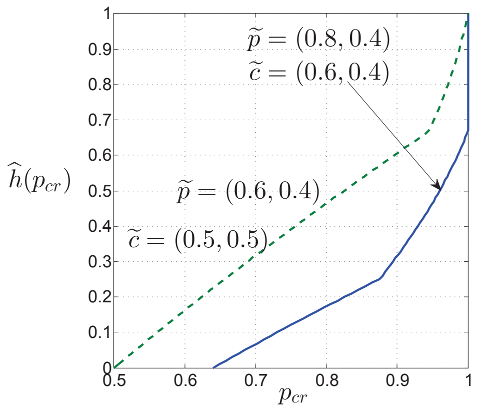

Figure 19: Robustness function from eq.(132) for the two options in eqs.(135) and (136).