89. **Uncertain linear elasticity** (based on exam 035018, 22.5.2016) (p.296). Consider a linear elastic system whose stress-strain relation is described by:

$$\varepsilon = \frac{\varepsilon_1}{\sigma_1}\sigma \quad \text{for} \quad 0 \le \sigma \tag{417}$$

The values of $\varepsilon_1$ and $\sigma_1$ define the endpoint of the linear-elastic domain in an idealized elasto-plastic model, though we will employ this model for all positive values of stress, $\sigma$.

(a) The values of $\varepsilon_1$ and $\sigma_1$ are uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ (\varepsilon_1, \sigma_1) :\ \varepsilon_1 \ge 0,\ \left| \frac{\varepsilon_1 - \widetilde{\varepsilon}_1}{\widetilde{\varepsilon}_1} \right| \le h,\ \sigma_1 \ge 0,\ \left| \frac{\sigma_1 - \widetilde{\sigma}_1}{\widetilde{\sigma}_1} \right| \le h \right\}, \quad h \ge 0 \tag{418}$$

where $\widetilde{\varepsilon}_1$ and $\widetilde{\sigma}_1$ are known and positive. A known positive stress, $\sigma_0$, will be applied, and we require that the strain not exceed the value $\varepsilon_0$. Derive an explicit algebraic expression for the robustness function.

(b) Return to the linear elastic model in eq.(417) and suppose the $\varepsilon_1$ is known but $\sigma_1$ is a random variable with an exponential distribution:

$$p(\sigma_1) = \lambda \mathrm{e}^{-\lambda \sigma_1}, \quad \sigma_1 \ge 0 \tag{419}$$

As before, we require that the strain not exceed the value $\varepsilon_0$. Derive an explicit algebraic expression for the probability of failure, namely, the probability that the strain exceeds $\varepsilon_0$.

(c) Continuing from part 89b, suppose that the threshold for mechanical failure, $\varepsilon_0$, is uncertain as represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon_0 :\ \varepsilon_0 \ge 0,\ \left| \frac{\varepsilon_0 - \widetilde{\varepsilon}_0}{s} \right| \le h \right\}, \quad h \ge 0 \tag{420}$$

We require that the probability of failure not exceed the critical value $P_{\mathrm{c}}$, which is strictly less than 1. Derive an explicit algebraic expression for the robustness function for this probabilistic requirement.

(d) Return to part 89a and derive an explicit algebraic expression for the opportuneness function, if we aspire to achieve a strain that is at least as small as $\varepsilon_w$, which is strictly greater than zero.

(e) Let us continue with eq.(417) but assume that $\varepsilon_1$ and $\sigma_1$ are known. However, the stress $\sigma$ is the result of a vector of forces, $f$, acting on the body:

$$\sigma = \psi^T f \tag{421}$$

where $\psi$ is a known vector. The force vector, $f$, is uncertain, as described by this ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ f :\ \left(f - \widetilde{f}\right)^T W^{-1} \left(f - \widetilde{f}\right) \le h^2 \right\}, \quad h \ge 0 \tag{422}$$

where $\widetilde{f}$ and $W$ are known and $W$ is a real, symmetric, positive definite matrix. We require that the strain not exceed the critical value $\varepsilon_0$. Derive an explicit algebraic expression for the robustness function.

(f) ‡ We now modify the stress-strain relation in eq.(417) by delimiting the range of validity of the linear relation:

$$\varepsilon = \frac{\varepsilon_1}{\sigma_1}\sigma \quad \text{for} \quad 0 \le \sigma \le \sigma_1 \tag{423}$$

The values of $\varepsilon_1$ and $\sigma_1$ define the endpoint of the linear-elastic domain in an idealized elasto-plastic model. However, the values of $\varepsilon_1$ and $\sigma_1$ are uncertain as specified in the info-gap model of eq.(418) which we now denote $\mathcal{U}_1(h)$. Hence, we do not know the upper limit, $\sigma_1$, of the domain of applicability of the linear relation. Let us suppose additional information about the stress-strain relation. Specifically, for $\sigma > \sigma_1$, the fractional error of the true strain function, $\varepsilon(\sigma)$, with respect to the linear model in eq.(417), is unknown:

$$\left| \varepsilon(\sigma) - \frac{\varepsilon_1}{\sigma_1}\sigma \right| \le \frac{\varepsilon_1}{\sigma_1}\sigma h, \quad \text{for } \sigma > \sigma_1 \tag{424}$$

We now formulate the overall info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon(\sigma) : \text{ for } 0 \le \sigma \le \sigma_1 : \ \varepsilon(\sigma) = \frac{\varepsilon_1}{\sigma_1}\sigma, \ \ (\varepsilon_1, \sigma_1) \in \mathcal{U}_1(h) \right.$$
$$\left. \text{for } \sigma_1 < \sigma : \ \left| \varepsilon(\sigma) - \frac{\varepsilon_1}{\sigma_1}\sigma \right| \le \frac{\varepsilon_1}{\sigma_1}\sigma h \right\} \tag{425}$$

Derive the inverse of the robustness function for the requirement that the strain not exceed the critical value $\varepsilon_0$ with known positive applied stress $\sigma_0$.