An Info-Gap Approach to Policy Selection for Bio-Terror Response

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Abstract. Bio-terror events are accompanied by severe uncertainty: great disparity between the best available data and models, and the actual course of events. We model this uncertainty with non-probabilistic information-gap models of uncertainty. This paper focuses on info-gaps in epidemiological models, in particular, info-gaps in the rate of infection. robustness to uncertainty is defined as a function of the required critical morbidity resulting from the attack. We show how preferences among available interventions are deduced from the robustness function. We demonstrate the irrevocable trade-off between robustness and demanded performance, and show that best-estimated performance has zero robustness. Finally, we present a theorem concerning the reversal of preferences between available interventions, and illustrate it with a numerical example.

1 Introduction and Motivation

The constant threat of bio-terror attacks on civilian populations in recent years has emphasized the importance of mathematical models as policy planning tools [1] – [5]. Due to the paucity of accurate scientific knowledge about the mechanisms underlying the dynamics of bio-terror events, attempts to model them will inevitably include a wide range of hypothetical premises and assumptions about the unknown. A careful decision maker should therefore treat results that are provided by conventional mathematical models skeptically, keeping in mind that no model accurately reflects unknown future occurrences, even more so due to the severe lack of reliable information.

When developing models to support a decision process, the uncertainties — the information gaps — should be carefully identified and incorporated into the models. The aim of this paper is to present a new methodology, info-gap decision theory, in mathematical modeling of highly uncertain phenomena, such as the aftermath of a bio-terror attack. Info-gap theory provides a new option-ranking criterion that allows the decision makers to evaluate the robustness of

their decisions to uncertainty, and also to assess the possibility of better than anticipated outcomes that the uncertainty may favor us with.

Probabilistic and statistical tools are suitable when the underlying processes are understood and stable. Info-gap decision theory, on the contrary, is most useful when decisions of high significance must be made in unique situations of high uncertainty.

2 Epidemic Model from an Info-Gap Point of View

Info-gap theory was invented and developed to provide decision makers with analytical tools that help establish preferences and to assess risks and opportunities when there is a severe lack of reliable information [6]. To outline the basic principles of info-gap theory and its applications in the field of defense against bioterrorism, we consider a simple epidemic model adopted from [7].

The model represents a spread of some highly infectious but not lethal disease, assuming that the total population of size N consists of susceptibles, $x(t)$ and infectives, $y(t)$. Infectives remain in contact with susceptibles for all time $t \geq 0$. Once a susceptible is infected, he becomes an infective and remains in that state. The susceptibles and infectives mix homogeneously with rate β, and in absence of intervention, according to this model, all the individuals in the population eventually become infected (at $t = \infty$ in equation (3)).

The equations describing the epidemic are:

$$x(t) + y(t) = N, t \geq 0 \tag{1}$$

$$\frac{dy}{dt} = \beta xy = \beta(N - y)y \tag{2}$$

were β and N are constant. Solving equation (2) we get:

$$y(t, \beta) = \frac{y_0 N}{y_0 + (N - y_0)\exp(-\beta N t)} \tag{3}$$

were y_o is the initial size of infected population at the beginning of the event, i.e at time $t = 0$.

In this simple model we view the β and N as decision parameters, i.e. response strategies may be formulated in terms of their influence on β and N. For example, timely announcement of an event may, for better or worse, affect the infection rate β. Different quarantine or vaccination policies may regulate the population size, N. The actual impact of policy interventions on outcomes is highly uncertain. By defining the notions of robustness of uncertain effects on β and N, we will establish the preference ranking on the available policies.

3 Modeling the Uncertainty

A wise decision making process not only should rely upon available knowledge, but also must not ignore the absence of information, as if all we know is all

there is. Info-gap theory views uncertainty as an unbounded family of nested sets of all possible events, and supplies a mechanism of incorporating this into the mathematical models.

To illustrate this we formulate here a simple model of uncertainty in the pair-wise rate of infection β.

Say we have a conjecture that the nominal pair-wise rate of infection varies over time in the following manner: It has a constant value, until time t_a, at which the announcement of a terrorist attack in public and other possible preventive measures take place. As a result the pair-wise rate of infection is expected to decline, but in an uncertain manner, i.e.:

$$\bar{\beta}(t) = \begin{cases} \bar{\beta} & \text{if } 0 \leq t \leq t_a \\ \bar{\beta} + \bar{\delta}t & \text{else} \end{cases} \tag{4}$$

and $\bar{\beta}$ and $\bar{\delta} < 0$ are constant. Since the real infection rate $\beta(t)$ is not known, we assume that it may differ from the conjectured rate of infection $\bar{\beta}(t)$, such that the relative error does not exceed some unknown "horizon" of uncertainty $\alpha \geq 0$. Here α is unbounded in its value, and thus admits all possible relative errors. The appropriate info-gap model would be an unbounded family of nested sets:

$$U(\alpha, \bar{\beta}(t)) = \left\{ \beta(t) : \left| \frac{\beta(t) - \bar{\beta}(t)}{\bar{\beta}(t)} \right| \leq \alpha, \beta(t) > 0 \right\}, \alpha \geq 0. \tag{5}$$

Note that if we take $t_a = \infty$, i.e. we don't announce the attack, we get that our conjectured infection rate is constant over time, $\bar{\beta}(t) = \bar{\beta}$.

4 Making Robust Decisions

Perfectly optimal solutions cannot be identified with highly imperfect models. We suggest looking for robust satisfactory decisions instead of optimal ones. But what is a robust decision? How does one measure robustness? Info-gap theory is well equipped to answer these questions. The evaluation of robustness to uncertainty is done by the means of the *robustness function* that expresses the greatest level of uncertainty at which failure cannot occur.

To be more specific, consider the demand that the size of the infected population, $y(t_c, \beta(t))$, at some time t_c such that $t_a < t_c$, does not exceed some critical value Y_c. Solving equation (2) with $\beta = \beta(t)$ and using our nominal function $\bar{\beta}(t)$ we may estimate that at time t_c infected population size is:

$$y(t_c, \bar{\beta}(t)) = \frac{y_0 N}{y_0 + (N - y_0) \exp\left[-N(\bar{\beta}t_c - \bar{\delta}(t_c^2 - t_a^2)/2)\right]} \tag{6}$$

Since the function $\bar{\beta}(t)$ is most possibly wrong, the result in equation (6) is hardly reliable and therefore is not a good basis for choosing policy interventions which influence $\bar{\beta}(t)$ or N. In the following subsections we introduce the notion of robustness to uncertainty in $\bar{\beta}(t)$ by defining and calculating the robustness

function. Through this example we show how the robustness function enables us to measure the reliability of policy interventions, make the adequate preferences and view the unavoidable trade-offs in demands and expectations vs. robustness to uncertainty.

4.1 Definition and Calculation of the Robustness Function

The robustness to uncertainty in $\bar{\beta}(t)$ is the greatest horizon of uncertainty α, up to which all possible functions $\beta(t)$ satisfy the requirement that at time t_c the infected population size is not greater than Y_c:

$$\widehat{\alpha}(Y_c, \bar{\beta}(t)) = \max \left\{ \alpha : \left(\max_{\beta(t) \in U(\alpha, \bar{\beta}(t))} y(t_c, \beta(t)) \right) \leq Y_c \right\} \tag{7}$$

After some calculations we get:

$$\widehat{\alpha}(Y_c, \bar{\beta}(t)) = \begin{cases} 0 & , 0 \leq Y_c \leq y(t_c, \bar{\beta}(t)) \\ \frac{1}{N(\bar{\beta} t_c - \bar{\delta}(t_c^2 - t_a^2)/2)} \ln\left(\frac{Y_c(N-y_0)}{y_0(N-Y_c)}\right) - 1 & , y(t_c, \beta(t)) < Y_c < N \\ \infty & , N \leq Y_c \end{cases} \tag{8}$$

where $t_a \leq t_c$.

It is easily seen from equation (8) that decreasing t_a we increase the value of robustness function, confirming our intuition that the earlier we intervene the greater would be our chances that at time t_c the size of infected population will not exceed some critical infection volume Y_c.

4.2 Basic Properties and Applications of the Robustness Function

Here we outline the basic properties of robustness function, which are also clearly seen from the graphs of robustness functions, in Fig. 1:

Substituting $Y_c = y(t_c, \bar{\beta}(t))$ in equation (8) we get

$$\widehat{\alpha}(y(t_c, \bar{\beta}(t)), \bar{\beta}(t)) = 0 \tag{9}$$

This means that obtaining morbidity as low as the estimated value, $y(t_c, \bar{\beta}(t))$, has no robustness to error in the estimate $\bar{\beta}(t)$. This result is very natural: since the parameter $\bar{\beta}(t)$ is an approximation, and sometimes even a guess, we cannot expect the prediction $y(t_c, \bar{\beta}(t))$ to be reliably realized.

The robustness function $\widehat{\alpha}(Y_c, \bar{\beta}(t))$ is increasing as a function of Y_c and $\lim_{Y_c \to N} \widehat{\alpha}(Y_c, \bar{\beta}(t)) = \infty$. This suggests that in order to gain robustness, one must accept poorer outcomes, i.e. to be content with higher values of critical infection Y_c. This property expresses the trade-off between demands and robustness to uncertainty: If high demands (small Y_c) are required then only low robustness (small $\widehat{\alpha}$) can be achieved.

If we view $\widehat{\alpha}$ at fixed critical infection volume Y_c, we can choose strategies that effect the parameters $N, \bar{\beta}$ and t_a in such a way that they maximize the

robustness function. (Actually this is exactly what we argued in (2) when suggesting early announcement.)

When we have several strategies that we need to compare we may use the robustness function to establish the necessary preference. Obviously we would prefer policies that imply greater robustness to uncertainty. In (4.3) we discuss in more detail the essence of this preference ranking.

4.3 Crossing of Robustness Curves for Constant β

In this section we will consider the case when the epidemic is not announced, i.e. $\bar{\beta}(t) = \bar{\beta}$. One of the essential parts in info-gap analysis is the establishment of existence of crossing of different robustness functions (as functions of Y_c and different values of decision parameters). If two robustness curves cross at some value y^* of Y_c then we have a reverse of preferences: Decisions that had high robustness at higher demands, $Y_c < y^*$, become less robust at weaker demands, $Y_c > y^*$. To illustrate this we state without proof here a proposition that assures existence of crossing for robustness functions in our simple epidemic model. We then discuss the possible implications.

Proposition 1. *For any given pair* $(\bar{\beta}_1, N_1)$ *there exist a pair* $(\bar{\beta}_2, N_2)$ *and a value* y^* *such that* $y_c(\bar{\beta}_1, N_1) < y_c(\bar{\beta}_2, N_2) < y^* < N_2 < N_1$ *and* $\widehat{\alpha}(y^*, \bar{\beta}_1, N_1) = \widehat{\alpha}(y^*, \bar{\beta}_2, N_2)$.

Consider a bio-terror attack on a city with a population size N_1 and initial estimated infection rate estimate $\bar{\beta}_1$. We can study our preferences among available interventions based on proposition 1 which is illustrated in fig. 2. Let $y^\star$ denote the infection volume at which the robustness curves cross, as illustrated in the figure. If we are willing to accept an infection volume larger than $y^\star$ at time t_c, then fig. 2 shows that we are more robust with interventions which are manifested in $\bar{\beta}_2$ and N_2. Greater robustness implies greater preference, so we prefer $\bar{\beta}_2$ and N_2 over $\bar{\beta}_1$ and N_1. Note that this preference is different from the preference based on the best-estimated infection rates.

On the other hand, if we require that the infection volume at time t_c be less than $y^\star$, then we are more robust with interventions $\bar{\beta}_1$ and N_1 and hence prefer the corresponding interventions. Succinctly:

$$(\bar{\beta}_2,\ N_2) \succ (\bar{\beta}_1,\ N_1) \quad \text{if and only if} \quad Y_c > y^\star \tag{10}$$

We note, however, that robustness trades-off against performance, implying that performance-requirement as strict as $Y_c < y^\star$ entails very low robustness.

Alternatively, we can establish preferences in terms of required robustness rather than required infection volume. Let $\widehat{\alpha}^\star$ denote the value of robustness at which the curves cross in fig. 2. If robustness at least as large as $\widehat{\alpha}^\star$ is required, then we prefer $\bar{\beta}_2$ and N_2; otherwise we prefer $\bar{\beta}_1$ and N_1.

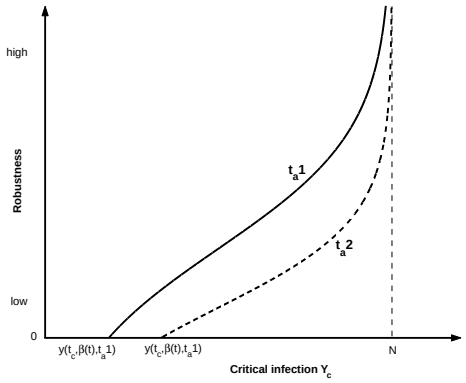

Fig. 1. Robustness function versus critical infection volume. Comparing different announcement times $t_a1 < t_a2$

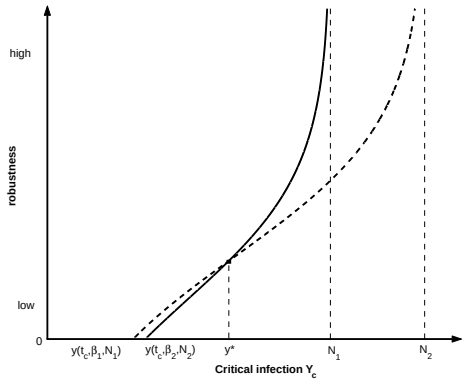

Fig. 2. Robustness function versus critical infection volume. Comparing different decision parameters $(\bar{\beta}, N)$

5 Future Research

A natural extension to our work will be moving from a simple epidemic model, which is illustrative but not realistic, to a more hard-nosed epidemiological model. In particular we intend to apply the Info-gap methodology to study the model presented by Kaplan and Wein [1].

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